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ON BETHE STRINGS IN THE TWO-PARTICLE SECTOR OF THE CLOSED $SU(2)_Q$ INVARIANT SPIN CHAIN

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In this paper we investigate complex solutions of the Bethe equations in the two-particle sector, both for arbitrary finite number of sites and for the thermodynamic limit. We find the number of complex solutions (strings) and compare it with the string conjecture prediction. Some simple properties of these solutions like position in the spectrum, crossing of levels, connection to the ground state and transformation to the real solutions are discussed. Counting both real and complex solutions, we find expected number of highest-weight Bethe states.

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1. *Introduction*

Among integrable spin chains, those invariant on quantum groups have attractedconsiderable interest. The simplest among these are $SU(2)_q$ invariant chains. Open chains have been considered for spin one half [1], spin one [2] and higher spin [3]. Generalisations to other groups have also been investigated [4,5]. Closed chains have been introduced later because parallel requirement of quantum-group invariance and generalised translational invariance required introduction of a nonlocal term in the Hamiltonian [6]. These chains were shown to have interesting properties. $SU(2)_q$ invariant closed chain has ground state with vanishing or nonvanishing spin depending on the value of the coupling constant [7]. Central charge was found and it was shown that in particular points of the coupling constant it corresponds to central charge of minimal unitary series [7,8]. Its excited states and operator content was also found [9]. Recently, it was argued that this model is related to interesting field theories [10] and in particular to the Liouville theory with imaginary coupling [11]. A common approach to investigation of integrable models in general, and of this model in particular, is the Bethe Ansatz method [12,13]. This method leads to a set of transcendental equations for momenta of quasiparticles. One class of the solutions has real quasimomenta that can be found by numerical iteration. However, there are also solutions with complex quasimomenta. Searching for the latter class of solutions, one usually makes the so-called string conjecture [14,15]. In particular, it has been used in investigations of some quantum invariant models [16,17] and also in a recent investigation motivated by the relation of such models to Liouville field theories[18]. However, it was recently pointed in the context of XXX and XXZ models that string conjecture has exceptions [19–25]. That means, those solutions to the Bethe Ansatz equations are not yet completely understood. For that reason, in this paper we investigate complex solutions for the closed quantum-invariant spin chain. In this investigation we shall not use the string conjecture.

For simplicity, we shall investigate the sector where spin is lower by two units than the maximal spin. We shall find the number of complex solutions for arbitrary number of sites N. In the thermodynamic limit, this number is found to be of the order N, as expected on the basis of the string conjecture. Next correction will differ for a finite number from the string-conjecture prediction. If the coupling constant tends to the Bethe point ($\cos\phi \to 1$) in a sufficiently fast manner, then we shall have an infinite number of exceptions to the number predicted by the string conjecture. The model in this point coincides with the XXX model and this is consistent with the results of Refs. [22] and [23]. Further, we shall find that the energy distribution of complex solutions show simple features and that they are located in a narrow energy band. In particular, they are located on the top of the spectrum near the antiferromagnetic point and on the bottom near the ferromagnetic point. Energy levels cross each other as coupling constant changes. There are no such crossings in the XXZ model. We also find that solutions of the one of the two classes of bound states (strings) evolve in real solutions in the special points for the coupling constant, where we know that the representation theory is not isomorphic to usual SU(2). It will turn also that at least one of overall ground states evolves in the string state. This happens in a region of coupling constant where it is not more a ground state.

As already anticipated, we shall consider the Hamiltonian

$$H = Nq - \sum_{i=1}^{N-1} R_i - R_0 \tag{1}$$

$$R_0 = G R_{N-1} G^{-1} \tag{2}$$

$$G = R_1 R_2 .... R_{N-1}\,, \tag{3}$$

where R_i are 4×4 matrices given by

$$R_i = \sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+ + \frac{q+q^{-1}}{2} \left(\sigma_i^3 \sigma_{i+1}^3 + 1\right) - \frac{q-q^{-1}}{4} \left(\sigma_i^3 - \sigma_{i+1}^3 - 2\right) . \quad (4)$$

Here q denotes a parameter which lies on unit circle

$$q = e^{\mathrm{i}\phi} , \quad (5)$$

and ϕ will be called coupling constant. The operators R_i satisfy the Hecke algebra

$$R_i^2 = \left(q - q^{-1}\right) R_i + 1 \quad (6)$$

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} . \quad (7)$$

As a consequence,

$$[G, H] = 0 . \quad (8)$$

It was also shown that this Hamiltonian is invariant on the $\mathrm{SU}(2)_q$ symmetry group whose generators are given by

$$S^3 = \frac{1}{2} \sum_{i=1}^{L} 1 \otimes \ldots \otimes \sigma_i^3 \ldots \otimes 1 \quad (9)$$

$$S^+ = \sum_{i=1}^{L} q^{-\sigma^3/2} \otimes ....\sigma_i^+ \otimes .... \otimes q^{\sigma^3/2} . \quad (10)$$

$$[S^+, S^-] = \frac{q^{2S^3} - q^{-2S^3}}{q - q^{-1}} . \quad (11)$$

Hamiltonian of the model is highly nonlocal, but due to the Hecke algebra, it is still integrable. It can be diagonalized, e.g., with the coordinate Bethe Ansatz method [7]. The energy eigenfunctions of the spin s

$$s = \frac{N}{2} - M, \quad (12)$$

with M spins down can be written as

$$|\psi_M\rangle = \sum_{1 \leq n_1 \leq n_2 \ldots \leq n_M \leq N} \psi_M \left(n_1, \ldots n_M\right) |n_1 \ldots n_M\rangle . \quad (13)$$

The ψ_M functions are given by

$$\psi_M(n_1, \dots n_M) = \sum_P \exp\left[\mathrm{i}\left(\sum_{j=1}^{M} k_{P_j} n_j + \frac{1}{2} \sum_{1\le j\le l\le M} \Phi_{P_j,P_l}\right)\right]. \tag{14}$$

Here the sum runs over the elements of the permutation group S_M. The phase shifts $\Phi_{j,i}$ have the following simple expression

$$\Phi_{j,i} = 2 \arctan \frac{\cos\phi \sin[(k_j - k_i)/2]}{\cos[(k_j + k_i)/2] - \cos\phi \cos[(k_j - k_i)/2]}. \tag{15}$$

The quasimomenta k_i, $i = 1 \dots M$ form a solution of the Bethe Ansatz equations

$$N k_i + \phi\,(2M - N - 2) + \sum_{j=1}^{M} \Phi_{i,j} = 2\pi\lambda_i \quad i = 1 \dots M. \tag{16}$$

The M Bethe numbers $I_i, i = 1 \dots M$ are half integers (integers) for M even (odd). In terms of quasimomenta $k_i, i = 1 \dots M$, the energy E and generalized momentum P read

$$E = 2 \sum_{i=1}^{M} (\cos\phi - \cos k_i)\,, \tag{17}$$

$$P = \sum_{i=1}^{M} k_i - \phi(N - M - 1)\,. \tag{18}$$

Operator G is then

$$G = e^{-\mathrm{i}P}\,. \tag{19}$$

Due to $\mathrm{SU}(2)_q$ symmetry, one can use its representation theory to classify the states. In fact, for generic q we have the same multiplet structure as for usual SU(2). However, for

$$q^p = \pm 1 \qquad\qquad p \quad \text{integer}, \tag{20}$$

additional degeneracy occurs [1,6]. In particular, representations of spins $j' = j + np$ and $j' = p - 1 - j - np$ mix. Here n is an integer. In order to get representations isomorphic to SU(2), one has to exclude also

$$j = np - 1/2\,. \tag{21}$$

Only the remaining representations (called 'good' representations) with

$$j < \frac{p-1}{2} \tag{22}$$

are isomorphic to SU(2). The parameter q, which is a root of unity, can be written as

$$q = e^{\mathrm{i}\pi p'/p} . \tag{23}$$

It has been shown [1] that for $p' = 1$, the representations are unitary. We shall find later that the points defined by (23) will play a role in the evolution of complex solutions into real ones when we vary the coupling constant. Let us for the moment concentrate on the generic sector of the parameter q. Due to the already mentioned property that in this case we have the usual SU(2) multiplet structure, it is sufficient to find the highest-weight states. Other states can be constructed [7] by application of the generator J^-. It is known that the highest-weight states correspond to sets of $\{k^i\}, i = 1 \ldots M$ where the quasimomenta satisfy

$$k_i \neq \phi . \tag{24}$$

In fact, we can identify in advance Bethe numbers which lead to the non-highest-weight solutions. Let us assume that for some $i = l$ we have

$$k_l = \phi . \tag{25}$$

A straightforward calculation using (15) shows that

$$\Phi_{l,j} \left(k_l = \phi, k_j \right) = \pi - 2\phi. \tag{26}$$

From the Bethe equation (16) for $i = l$, we obtain

$$\lambda_l = \frac{M-1}{2} . \tag{27}$$

The highest-weight solutions will be obtained by excluding (27) from the choice of the Bethe numbers. The ground state for the Hamiltonian (1) was found for the whole interval $0 \leq \phi \leq \pi$. In fact, contrary to the XXZ model, the spin zero state is the ground state only in the $\frac{\pi}{2} \leq \phi \leq \pi$ region. In the rest of the interval, there is a subregion for each spin, where the ground state has just that particular spin. More precisely, the total spin s of the ground state depends on the coupling constant ϕ according to

$$\begin{gathered} J = 0 \quad \text{for} \quad \frac{\pi}{2} \leq \phi \leq \pi \\ J = s \quad \text{for} \quad \frac{\pi}{2(s+1)} \leq \phi \leq \frac{\pi}{2s} \\ J = \frac{N}{2} \quad \text{for} \quad 0 \leq \phi \leq \frac{\pi}{N}. \end{gathered} \tag{28}$$

The Bethe numbers which give ground state are given by

$$\lambda = \left(-\frac{M-1}{2} \right) - 1, \ldots, \left(\frac{M-1}{2} \right) - 1 \tag{29}$$

and

$$M = \frac{N}{2} - s\,. \tag{30}$$

It also turns out that the corresponding momenta are real. The natural question arises, which role play complex solutions of the Bethe equations.

2. *Complex solutions in two particle sector*

In this section we shall investigate complex solutions of the Bethe equations (16) without assuming the string conjecture [14,15]. For simplicity, we shall work in the $M = 2$ sector. In that case, the Bethe equations (16) take the form

$$Nk_1 + \Phi_{1,2} + \phi(2-N) = 2\pi\lambda_1, \tag{31}$$

$$Nk_2 - \Phi_{1,2} + \phi(2-N) = 2\pi\lambda_2. \tag{32}$$

Here we want to look in particular for complex solutions. Due to reality of energy and generalized momentum, k_1 and k_2 have to be complex conjugates of each other

$$k_1 = k_\mathrm{r} + \mathrm{i}k_\mathrm{i}, \tag{33}$$

$$k_2 = k_\mathrm{r} - \mathrm{i}k_\mathrm{i}. \tag{34}$$

We can express k_r and k_i by taking sum and difference of equations (31) and (32)

$$k_\mathrm{r} = \frac{1}{N}\left[\pi(\lambda_1+\lambda_2) - \phi(2-N)\right] \tag{35}$$

$$\mathrm{i}Nk_\mathrm{i} = \pi(\lambda_1-\lambda_2) - 2\arctan\frac{\cos\phi\sin(\mathrm{i}k_\mathrm{i})}{\cos k_\mathrm{r} - \cos\phi\cos(\mathrm{i}k_\mathrm{i})}. \tag{36}$$

With the help of the identity

$$\arctan z = \frac{1}{2\mathrm{i}}\ln\frac{1+\mathrm{i}z}{1-\mathrm{i}z} \tag{37}$$

and exponentiation of (36), one can obtain

$$\frac{\sinh[k_\mathrm{i}((N/2)-1)]}{\sinh(k_\mathrm{i}N/2)} = \frac{\cos k_\mathrm{r}}{\cos\phi} \qquad \lambda_1+\lambda_2 \quad \mathrm{odd}\,, \tag{38}$$

$$\frac{\cosh[k_\mathrm{i}((N/2)-1)]}{\cosh(k_\mathrm{i}N/2)} = \frac{\cos k_\mathrm{r}}{\cos\phi} \qquad \lambda_1+\lambda_2 \quad \mathrm{even}. \tag{39}$$

In the further analysis, we shall call the solutions of equation (38) s-solutions and the solutions of equation (39) c-solutions. The left-hand sides of equations (38) and

(39) are monotonously decreasing functions, so we shall have a solution for k_i for any k_r for which $\cos k_\mathrm{r}$ is in the interval

$$\text{s-strings} \quad : \quad 0 \le \cos k_\mathrm{r} < \cos\phi(1 - \frac{2}{N}), \tag{40}$$

$$\text{c-strings} \quad : \quad 0 \le \cos k_\mathrm{r} < \cos\phi \tag{41}$$

if $\cos\phi \ge 0 \quad (0 \le \phi \le \frac{\pi}{2})$ and

$$\text{s-strings} \quad : \quad \cos\phi(1 - \frac{2}{N}) < \cos k_\mathrm{r} \le 0, \tag{42}$$

$$\text{c-strings} \quad : \quad \cos\phi < \cos k_\mathrm{r} \le 0 \tag{43}$$

if $\cos\phi \le 0 \quad (\frac{\pi}{2} \le \phi \le \pi)$. Further, as long as $k_\mathrm{i} \neq 0$, $k_{1,2} \neq \phi$ and, therefore, this solution represents the highest-weight state. Now, we can proceed to find the number of complex solutions. As a first step, we shall consider leading order in N when the inequalities (40) and (42) are identical. In this case, for a given coupling constant ϕ, the number of complex solutions will depend on the number of values that k_r can take. In order to have one to one correspondence between k_r and $\cos k_\mathrm{r}$, the sum of Bethe numbers can take $2N-1$ different equidistant values. The interval in k_r, for which there are complex solutions, is $2\left(\frac{\pi}{2} - \phi\right)$. As a result, the number of the string solutions in leading order in N is

$$\frac{1}{2\pi}(2N-1)(\pi - 2\phi). \tag{44}$$

If we insist to have k_r between π and $-\pi$, we should not take Bethe numbers symmetrically around zero due to the term proportional to ϕ in Eq. (35). However, this does not affect the counting argument. To determine the number of complex solutions more precisely, we have to take into account subleading orders in N in relations (40) and (42). The allowed interval for the real parts of s-solutions is smaller than the corresponding interval for c-solutions, and there are no s-solutions for an interval in k_r of length

$$\delta k_\mathrm{r} = 4 \arcsin \frac{\cos\phi}{N \sin\left\{\frac{1}{2}[\arccos\cos\phi + \arccos(\cos\phi(1-2/N))]\right\}}. \tag{45}$$

The number of solutions, which were overcounted in naive formula (44), is integer part of

$$n = \frac{2N-1}{\pi} \arcsin \frac{\cos\phi}{N \sin\left\{\frac{1}{2}[\arccos\cos\phi + \arccos(\cos\phi(1-2/N))]\right\}}. \tag{46}$$

This correction is a finite number even in the thermodynamic limit ($N \to \infty$). However, if $|\cos\phi| = 1$, correction is infinite and goes as $\sqrt{N}$. This is true if $|\cos\phi| \to 1$ sufficiently fast, more precisely if

$$N(1 - |\cos\phi|)^\alpha = \text{const.} \quad \text{and} \quad \alpha > 1. \tag{47}$$

This is consistent with the fact that $\phi = 0$ or $\phi = \pi$ corresponds to the XXX chain for which is known that this correction is infinite. This correction represents at the same time a violation of the string conjecture which was found previously in the XXX [22,23] and XXZ chains [24,25]. Disappearance of s-solutions (strings) is followed by appearance of real a solution with two close quasimomenta. These quasimomenta can be found near the number of sites where s-string disappeared by numerical iteration of (31) and (32) with the same Bethe numbers λ_1 and λ_2. Real solutions of Bethe equations with two identical Bethe numbers also represent violation of the string conjecture. Evolution of s-strings into real solutions with identical Bethe numbers can be followed for fixed coupling constant by increasing the number of sites N. Example is given in Fig. 1 for $\phi = 0.32$.

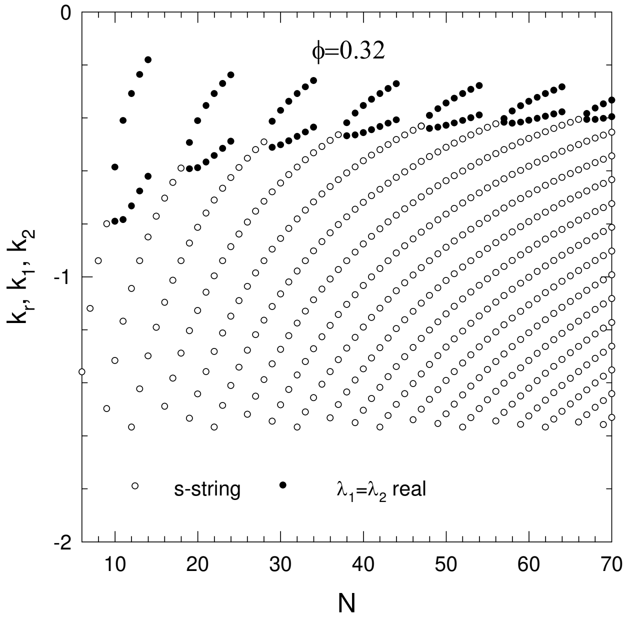

Fig. 1. This figure shows dependence of real part of complex s*-solutions on numbers of sites N for $\phi = 0.32$. It clearly illustrates transmutation of one complex solution in real solution (two quasimomenta) for a given critical N. These two real quasimomenta correspond to same Bethe numbers and are obtained by numerical iteration of equations (31) and (32).*

3. *Properties of complex solutions*

Next, the interesting question we would like to ask is how are complex solutions (bound states) distributed on the energy scale. From the relations (39), (38) and (17), we see that complex solutions are confined in the energy band

$$0 < E(\text{c-strings}) \leq 2\cos\phi, \tag{48}$$

$$\frac{8\cos\phi}{N} < E(\text{s-strings}) \leq 2\cos\phi. \tag{49}$$

The left sides of the inequalities correspond to the points in the coupling constant ϕ where two complex quasimomenta collide on the real axis and the complex solution becomes a real solution (decay of the bound state). The right sides of the inequalities correspond to points where $k_{\rm i}$ tends to infinity and so the localisation of two overturned spins in bound state tends to infinity. This can be seen from the form of the Bethe wave function (14). The energy band (48) is generally narrow compared to the overall spread of energy when all solutions are included. As an illustration of $M = 2$ spectrum, figures are given for $N = 6, N = 8$ and $N = 10$ (Figs. 2, 3 and 4, respectively). We see that all bound states (strings) disappear

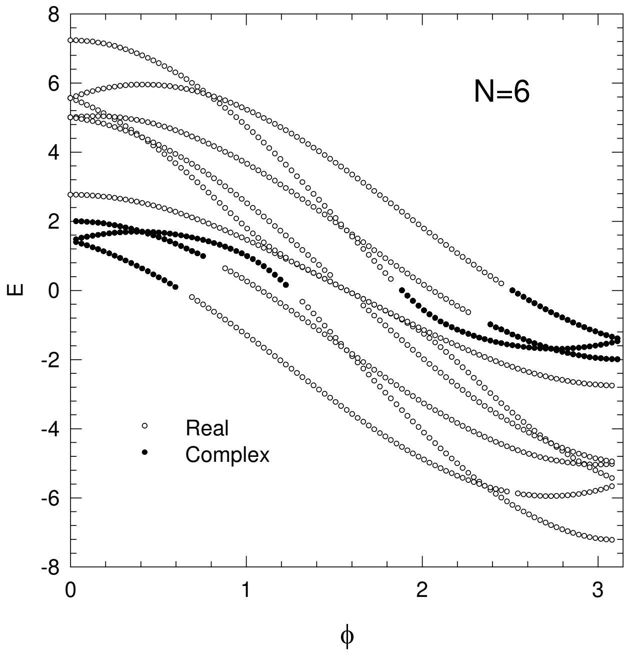

Fig. 2. Spectrum of $M = 2$ sector for $N = 6$. Energies of both real and complex solutions of the Bethe equations are plotted.

near the 'free theory' point $\phi = \frac{\pi}{2}$. Energies of the string solutions are on the top of the spectrum near the Bethe antiferromagnetic point $\phi = \pi$ and on the bottom of the spectrum near the Bethe ferromagnetic point $\phi = 0$. There is an interesting question connected to the nature of the overall ground state. We know that for any finite and even N, the total spin s of the ground state depends on the value of the coupling ϕ. In particular, ground state has spin zero for $\frac{\pi}{2} \leq \phi \leq \pi$ and spin s for $[\pi/(2(s+1)] \leq \phi \leq \pi/(2s)$. The quasimomenta are known and they are all real. We find that beyond these intervals, there is a region of ϕ values where these solutions become complex. To see this feature, we note that due to the relation $M = \frac{1}{2}N - s$

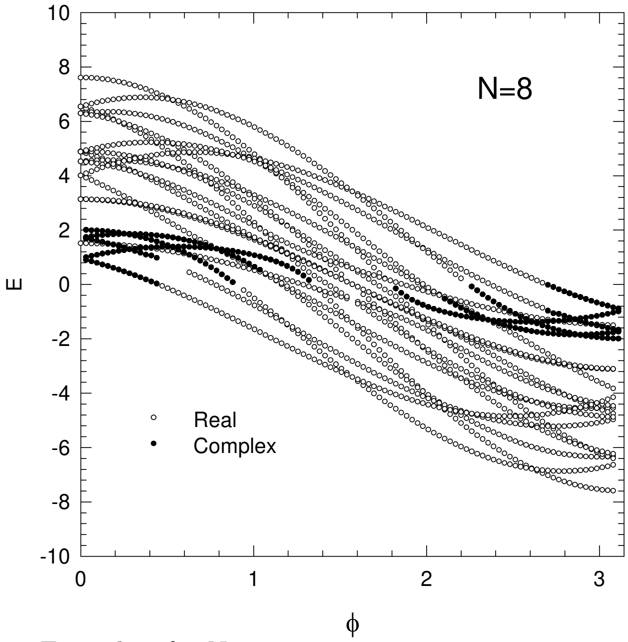

Fig. 3. Same as Fig. 2 but for $N = 8$.

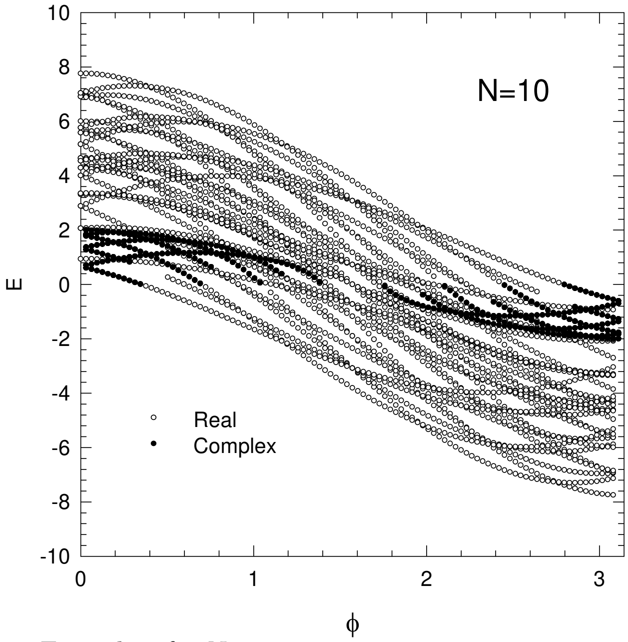

Fig. 4. Same as Fig. 2 but for $N = 10$.

and (28), the overall ground state is in the $M = 2$ sector for

$$\frac{\pi}{N-2} \leq \phi \leq \frac{\pi}{N-4}. \tag{50}$$

In particular, for $N = 6, 8$ and 10, these intervals in coupling will be $[\frac{\pi}{4}, \frac{\pi}{2}], [\frac{\pi}{6}, \frac{\pi}{4}]$ and $[\frac{\pi}{8}, \frac{\pi}{6}]$, respectively. Following this state, we see that at certain points outside the above intervals the quasimomenta become complex. From the choice of Bethe numbers (29) we see that this state will become complex c-solution and the transformation of this real state into string will happen at the $E = 0$ point. Generally, we can find points in the coupling constant when the complex solutions will become real. In the transition points

$$\cos\phi = \cos k_r \qquad \text{c-strings} \tag{51}$$

$$\cos\phi \left(1 - \frac{2}{N}\right) = \cos k_r \qquad \text{s-strings.} \tag{52}$$

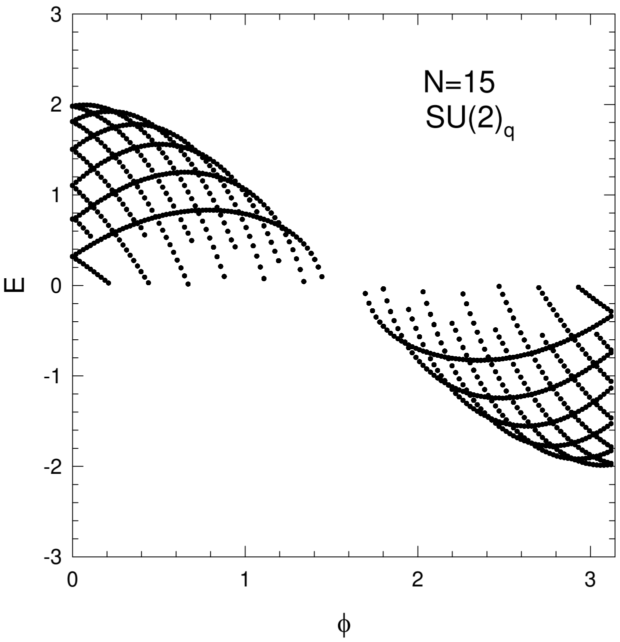

Fig. 5. Dependence of energies of complex solutions for $SU(2)_q$ invariant spin chain and number of sites $N = 15$ on coupling constant ϕ.

For c-strings, this is satisfied for

$$\phi = \frac{\pi}{2(N-1)}[2, ..., N-4, N-2] \qquad 0 \leq \phi \leq \frac{\pi}{2} \tag{53}$$

$$\phi = \frac{\pi}{2(N-1)}[N, ..., 2N-6, 2N-4] \qquad \frac{\pi}{2} \leq \phi \leq \pi. \tag{54}$$

Now we see that the state, which is the overall ground state in the interval (50), after a change of $\pi/[(N-1)(N-2)]$ in ϕ becomes c-string. It is interesting that the points (53) and (54) correspond to the points (23) where the representation theory is no more isomorphic to SU(2) and where several multiplets merge together forming indecomposable combinations [1]. The points where the s-strings disappear are not of this form because of the correction factor $(1 - 2/N)$ in (52).

Crossing of energy levels of complex solutions with the change of coupling constant is one of the features of this model that makes it different from the XXZ model. This is illustrated in Figs. 5 and 6 in which we follow energy levels of $M = 2$ strings for both models and for the number of sites $N = 15$. The number of strings is the same in both models. However, in XXZ model there are complex solutions with $\pm k_r$ which are degenerated. Presence of a term linear in ϕ in Eqs. (16), which can be interpreted as a coupling-constant-dependent toroidal twist in the XXZ model, removes this degeneracy and causes crossing of energy levels.

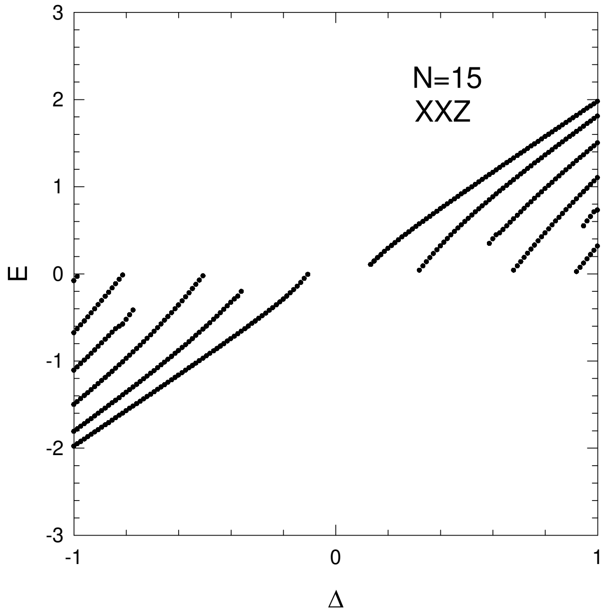

Fig. 6. Dependence of energies of complex solutions for XXZ spin chain and number of sites $N = 15$ on coupling constant Δ, which corresponds to $\cos\phi$.

4. *Real quasimomenta and completeness of solutions of Bethe equations*

We want to enumerate all real solutions of Bethe equations in the two-particle sector. We start again from Eqs. (31) and (32). After manipulating their difference and sum, we obtain the following equations for $k = k_1 + k_2$ and $k_1 - k_2$

$$\frac{k}{2} = \frac{1}{N}[\pi(\lambda_1 + \lambda_2) - \phi(2 - N)] \tag{55}$$

$$\frac{\sin\left(\frac{1}{2}(k_1 - k_2)(\frac{1}{2}N - 1)\right)}{\sin\left(\frac{1}{2}(k_1 - k_2)\frac{1}{2}N\right)} = \frac{\cos\frac{1}{2}k}{\cos\phi} \qquad \lambda_1 + \lambda_2 \quad \text{odd} \tag{56}$$

$$\frac{\cos\left(\frac{1}{2}(k_1 - k_2)(\frac{1}{2}N - 1)\right)}{\cos\left(\frac{1}{2}(k_1 - k_2)\frac{1}{2}N\right)} = \frac{\cos\frac{k}{2}}{\cos\phi} \qquad \lambda_1 + \lambda_2 \quad \text{even.} \tag{57}$$

One should notice that not all $2N-1$ different values of $I = \lambda_1+\lambda_2$ give different solutions. Replacing I by N is equivalent to changing one quasimomentum by 2π, which results in the change of sign of right-hand sides of (56) and (57). This reduces the number of possible values of I to N, which in turn can be chosen to give positive values of right-hand sides of (56) and (57). The left-hand sides of the Eqs. (56) and (57) are periodic functions. Thus in principle, for each of the N different fixed values of the right-hand side, one can count the number of solutions by counting the number of intersections. These periodic functions are given in Figs. 7 for $N = 9$ and $N = 10$. For N odd we find that the number of intersections equal is to

$$N_o^e = \frac{N-1}{2}\theta\left(\frac{\cos\frac{k}{2}}{\cos\phi} - 1\right) + \frac{N-3}{2}\theta\left(1 - \frac{\cos\frac{k}{2}}{\cos\phi}\right) \quad I \text{ even} \tag{58}$$

$$N_o^o = \frac{N-1}{2}\theta\left(\frac{\cos\frac{k}{2}}{\cos\phi} - 1 + \frac{2}{N}\right) + \frac{N-3}{2}\theta\left(1 - \frac{2}{N} - \frac{\cos\frac{k}{2}}{\cos\phi}\right) \quad I \text{ odd} \tag{59}$$

while for N even it is

$$N_e^e = \frac{N}{2}\theta\left(\frac{\cos\frac{k}{2}}{\cos\phi} - 1\right) + \frac{N-2}{2}\theta\left(1 - \frac{\cos\frac{k}{2}}{\cos\phi}\right) \quad I \text{ even} \tag{60}$$

$$N_e^o = \frac{N-2}{2}\theta\left(\frac{\cos\frac{k}{2}}{\cos\phi} - 1 + \frac{2}{N}\right) + \frac{N-4}{2}\theta\left(1 - \frac{2}{N} - \frac{\cos\frac{k}{2}}{\cos\phi}\right) \quad I \text{ odd.} \tag{61}$$

The number of complex solutions is just equal to the second θ function in the expressions above. If we take into account that for N even, there are $\frac{1}{2}N$ even and $\frac{1}{2}N$ odd values of I, and for N odd $\frac{1}{2}(N-1)$ even values of I and $\frac{1}{2}(N+1)$ odd values of I, we find that for any number of sites N and coupling constant ϕ there are

$$\frac{N(N-1)}{2} \tag{62}$$

solutions to the Bethe equations in the two-particle sector. Among these solutions, there are N previously identified non-highest-weight states, which have one of quasimomenta equal to ϕ. Finally, we obtain

$$\binom{N}{2} - \binom{N}{1} \tag{63}$$

highest-weight Bethe states.

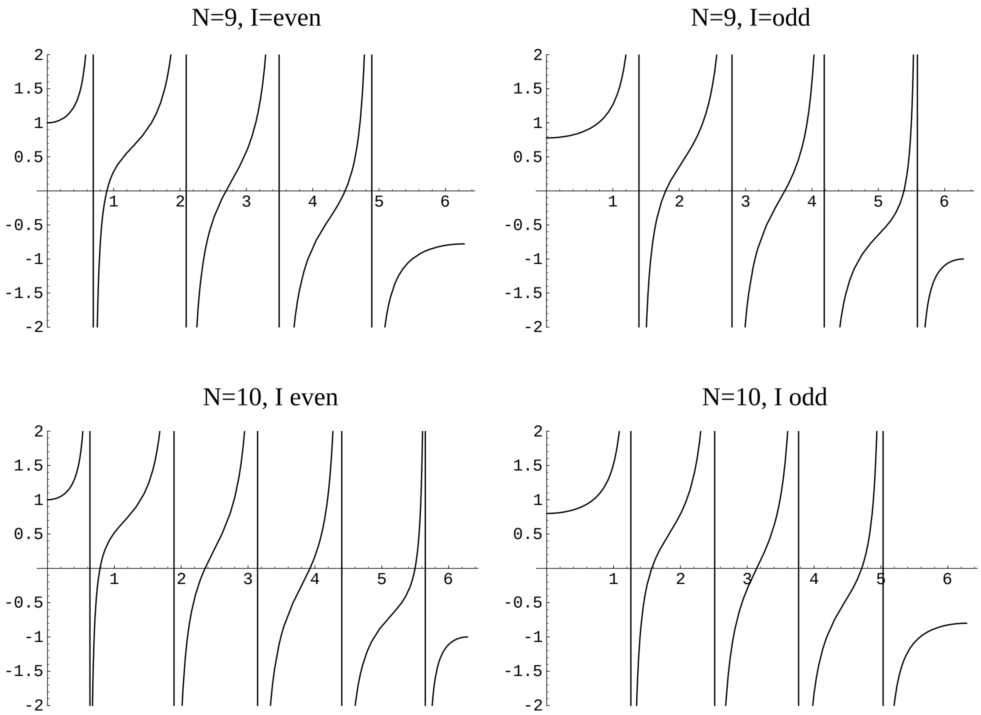

Fig. 7. Left hand sides of equations (56) and (57) as a functions of $k_1 - k_2$ for $N = 9$ and $N = 10$.

5. *Conclusion*

In this paper, we have investigated complex solutions of the Bethe equations for the $M = 2$ sector of the $\mathrm{SU}(2)_q$ invariant closed-spin chain for arbitrary number of sites and coupling constant. We find that some properties of these solutions are similar to the properties of complex solutions for the normal XXZ chain and some of them are not. In particular, we find the number of complex solutions both for a finite N and in thermodynamic limit. This number differs from the number predicted by the string conjecture for a finite number of solutions. However, if $\phi \to 0$

sufficiently fast compared to $1/N$, the number of exceptions becomes infinite and goes as $\sqrt{N}$. These properties are essentially the same as in the case of the XXZ chain. One can follow evolution of string solutions and their disappearance with decreasing 'coupling strength' $|\cos\phi|$. One class of strings (c-strings) disappears in points where the representation of the $\mathrm{SU}(2)_q$ is no more isomorphic to SU(2). By increasing the number of sites for fixed coupling constant, strings of the other class turn into real quasimomenta that can be found by iterating Bethe equations with two identical Bethe numbers. This is again violation of the string conjecture. The overall ground state, which is always real, is in the $M = 2$ sector for the coupling constant $\pi/(N-2) \leq \phi \leq \pi/(N-4)$. This state becomes a complex solution for $\phi \leq \pi/(N-1)$. The energy dependence of string solutions shows some simple features. The strings are found in a narrow energy band and are located on the top of the spectrum near the antiferromagnetic point and on the bottom near the ferromagnetic point. Their energy levels cross each other with the change of the coupling constant, which is not the case for the XXZ chain. Finally, we find the number of real solutions. The number of all solutions of the Bethe equations is $\binom{N}{2}$. Among these we identify N non-highest-weight states. That leads to the $\binom{N}{2} - \binom{N}{1}$ highest-weight Bethe states.

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BETHEOVE STRUNE U DVOČESTIČNOM SEKTORU ZATVORENOG $SU(2)_q$ INVARIJANTNOG SPINSKOG LANCA

Istražujemo kompleksna rješenja Betheovih jednadžbi u dvočestičnom sektoru za proizvoljan konačan broj kompleksnih rješenja i za termodinamičku granicu. Nalazimo broj kompleksnih rješenja (struna) i uspoređujemo ga s uobičajenim pretpostavkama za strune. Raspravljamo neka jednostavna svojstva tih rješenja, kao položaj u spektru, presjecanje stanja, spoj s osnovnim stanjem i preobrazbu k realnim rješenjima. Brojeći realna i kompleksna rješenja, našli smo očekivan broj Betheovih stanja najveće težine.