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LETTER TO THE EDITOR

RESTORATION OF CHIRAL SYMMETRY IN THE LINEAR SIGMA MODEL

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We study the chiral phase transition in the linear sigma model at nonzero temperature and baryon density with $N_f \geq 3$ quark flavours and N_c colours. One-loop calculations for $N_f = 3$ predict a first-order phase transition at both $\mu = 0$ and $\mu \neq 0$. We also compare the results with the already existing results for $N_f = 2$.

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Because of small quark masses, quantum chromodynamics (QCD) possesses the approximative chiral SU(N_f)×SU(N_f) symmetry for $N_f = 2, 3$. Since perturbative QCD cannot give good quantitative predictions at low energies, it is convenient to use various effective models. In this paper we study the chiral linear σ-model. In this model, the chiral symmetry is spontaneously broken, but it restores at high energies (i.e. temperatures). The aims of this paper are to describe the symmetry breaking at the classical level, to investigate the phase transition after which the symmetry restores and to determine the critical temperature and the order of the phase transition. The results for $N_f = 2$ have already been published [1]. Here we generalize some of the results to $N_f \geq 3$, with special emphasis on $N_f = 3$, and compare them with the results for $N_f = 2$.

Let us start with the σ-model at the classical level. Our approach is similar to the approach of Ref. 2. For $N_f \geq 3$, we introduce the matrix bosonic field

$$\Phi(x) = \sum_{a=0}^{N_f^2-1} (\sigma_a(x) + \mathrm{i}\pi_a(x)) \frac{\lambda_a}{\sqrt{2}} , \tag{1}$$

where $\lambda_0 = \sqrt{2/N_f}\, \mathbb{1}_{N_f \times N_f}$, whereas λ_a, $a = 1, 2, \ldots, N_f$, are the standard generators of the fundamental representation of the group SU(N_f). The hermitian fields π_a represent the light pseudoscalar mesons, while the hermitian fields σ_a represent the corresponding heavy scalar mesons.

The case $N_f = 2$ is special because the representations 2 and $\bar{2}$ of SU(2) are equivalent. An irreducible representation is provided by the choice

$$\Phi(x) = \frac{1}{\sqrt{2}}[\sigma(x)\mathbb{1}_{2\times 2} + i\boldsymbol{\pi}(x)\cdot\boldsymbol{\tau}] \; . \tag{2}$$

The most general Lagrangian constructed from the fields $\Phi(x)$ which is SU(N_f)×SU(N_f) invariant and renormalizable for any N_f, in the Euclidean notation takes the form

$$\mathcal{L}_b = \frac{1}{2}\,\mathrm{Tr}\,[\partial_\mu\Phi\partial_\mu\Phi^\dagger] + V(\Phi,\Phi^\dagger) \; , \tag{3}$$

where

$$V(\Phi,\Phi^\dagger) = \frac{m_0^2}{2}\,\mathrm{Tr}\,(\Phi\Phi^\dagger) + \frac{\lambda}{4}\,(\mathrm{Tr}\,\Phi\Phi^\dagger)^2 + \frac{\gamma}{4}\,\mathrm{Tr}\,(\Phi\Phi^\dagger)^2 \; . \tag{4}$$

The full symmetry of (3) is actually $\mathrm{U}_A(1)$×SU(N_f)×SU(N_f) [3]. For $N_f = 3$, one can also add a renormalizable term proportional to $\det\Phi + \det\Phi^\dagger$, whose symmetry is SU($N_f$)×SU($N_f$).

The interaction with fermions for $N_f \geq 3$ takes the form

$$\mathcal{L}_f = \bar{\psi}\left[\gamma_\mu\partial_\mu + g\sqrt{\frac{N_f}{2}}\sum_a(\sigma_a + \mathrm{i}\pi_a\gamma_5)\lambda_a\right]\psi \; , \tag{5}$$

whereas for $N_f = 2$ it becomes

$$\mathcal{L}_f = \bar{\psi}\,[\gamma_\mu\partial_\mu + g(\sigma + \mathrm{i}\boldsymbol{\pi}\cdot\boldsymbol{\tau}\gamma_5)]\,\psi \; . \tag{6}$$

The fermions in (5) and (6) are interpreted as constituent quarks with $N_c = 3$ colours [1,4,5].

Let us now consider the symmetry breaking at the classical level. The potential for the SU(3)×SU(3) σ-model is given by

$$V(\Phi,\Phi^\dagger) = \frac{m_0^2}{2}\,\mathrm{Tr}\,(\Phi\Phi^\dagger) + \frac{\lambda}{4}\,(\mathrm{Tr}\,\Phi\Phi^\dagger)^2 + \frac{\gamma}{4}\,\mathrm{Tr}\,(\Phi\Phi^\dagger)^2 - \tau(\det\Phi + \det\Phi^\dagger) \; . \tag{7}$$

The expectation value $\langle\Phi\rangle$ can be diagonalized and chosen to be real [6], so we find

$$\begin{aligned}
&\langle\Phi\rangle = \langle\Phi^\dagger\rangle \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{2}{3}}\langle\sigma_0\rangle + \langle\sigma_3\rangle + \frac{1}{\sqrt{3}}\langle\sigma_8\rangle & & \\ & \sqrt{\frac{2}{3}}\langle\sigma_0\rangle - \langle\sigma_3\rangle + \frac{1}{\sqrt{3}}\langle\sigma_8\rangle & \\ & & \sqrt{\frac{2}{3}}\langle\sigma_0\rangle - \frac{2}{\sqrt{3}}\langle\sigma_8\rangle \end{pmatrix} \\
&\equiv \begin{pmatrix} v_1 & & \\ & v_2 & \\ & & v_3 \end{pmatrix} , \qquad v_i \in \mathbf{R} . \qquad (8)
\end{aligned}$$

Therefore, the vacuum part of the potential (7) can be written as

$$V = \frac{m_0^2}{2} \sum_i v_i^2 + \frac{\lambda}{4} \left(\sum_i v_i^2 \right)^2 + \frac{\gamma}{4} \sum_i v_i^4 - 2\tau v_1 v_2 v_3 . \qquad (9)$$

The τ-term in (7) leads to a first-order phase transition and this term should be included if we want to incorporate the $\mathrm{U}_A(1)$ anomaly [3]. However, from now on we work with the approximation $\tau = 0$, because this choice simplifies the calculations and provides a unique theoretical framework for all numbers of flavours $N_f \geq 3$. It is also interesting to see whether the order of the transition will change by the omittion of the τ-term. Note also that the choice $\tau = 0$ does not violate the renormalizability, because the presence of τ changes the symmetry of the potential.

With $\tau = 0$, (9) is valid also for $N_f \geq 3$. The simplest minimum of the potential is obtained by putting $\langle\sigma_0\rangle \neq 0$, $\langle\sigma_i\rangle = 0$, $i = 1, 2, \ldots, N_f^2 - 1$, which means that all v_i are equal. This leads to

$$\langle\sigma_0\rangle^2 \equiv \sigma^2 = f_\pi^2 = \frac{-m_0^2}{\lambda + \frac{\gamma}{N_f}} . \qquad (10)$$

By the redefinition $\sigma_0(x) \to \sigma + \sigma_0'(x)$ in the Lagrangian, we find the masses

$$\begin{aligned}
m_{\vec{\sigma}}^2 &= m_0^2 + \left(\lambda + \frac{3\gamma}{N_f} \right) f_\pi^2 = \frac{2\gamma}{N_f} f_\pi^2 , \\
m_{\sigma_0}^2 &= m_0^2 + 3 \left(\lambda + \frac{\gamma}{N_f} \right) f_\pi^2 = 2 \left(\lambda + \frac{\gamma}{N_f} \right) f_\pi^2 , \\
m_\pi^2 &= m_0^2 + \left(\lambda + \frac{\gamma}{N_f} \right) f_\pi^2 = 0 , \\
m_F &= g\, f_\pi , \qquad (11)
\end{aligned}$$

where $\vec{\sigma} \equiv \{\sigma_1, \sigma_2, \ldots, \sigma_{N_f^2-1}\}$.

For $N_f = 2$, we take $\gamma = 0$, because $(\mathrm{Tr}\,\Phi\Phi^\dagger)^2 = (\sigma^2 + \boldsymbol{\pi}^2)^2 = 2\,\mathrm{Tr}\,(\Phi\Phi^\dagger)^2$. The $\vec{\sigma}$ degrees of freedom are absent for $N_f = 2$, because of (2). If we omit the first equation in (11), the relations (10) and (11) are valid also for $N_f = 2$ when $\gamma = 0$ [1]. In later numerical calculations we take $f_\pi = 92.4$ MeV, $m_\sigma = 1000$ MeV, $m_F = 340$ MeV, as in Ref. 1, and $m_{\vec{\sigma}} = 1300$ MeV.

Let us now study quantum and thermal fluctuations in the σ-model. This has already been studied for $N_f = 2$ in Ref. 1, where the saddle-point method has been used. It has also been shown in Ref. 1 that the saddle-point method is equivalent to the standard approach [7–9]. Technical details for $N_f \geq 3$ are similar to those for $N_f = 2$. Therefore, using the same notation as in Ref. 1, we just give the main results in the standard approach for $N_f \geq 3$. The results are equivalent to the results which would be obtained using the saddle-point method.

After the redefinition $\sigma_0(x) \to \sigma + \sigma_0'(x)$ in the Lagrangian, we find the effective potential

$$\Omega(\sigma, T, \mu) = \left(\lambda + \frac{\gamma}{N_f}\right)\frac{\sigma^4}{4} + \frac{m_0^2}{2}\sigma^2 + \Omega_0(\sigma, T, \mu) + \Omega_I(\sigma, T, \mu)\ , \qquad (12)$$

where

$$\Omega_0 = \Omega_F + \Omega_{\sigma_0} + \Omega_{\vec{\sigma}} + \Omega_\pi\ , \qquad (13)$$

$$\begin{aligned}
\Omega_F &= -N_c N_f \frac{1}{\beta}\sum_l \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\ \mathrm{Tr}\ \ln\left[\beta(-i\not{p} + m_F)\right]\ , \\
\Omega_{\sigma_0} &= \frac{1}{2\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\ \ln\left[\beta^2(k^2 + m_{\sigma_0}^2)\right]\ , \\
\Omega_{\vec{\sigma}} &= (N_f^2 - 1)\,\frac{1}{2\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\ \ln\left[\beta^2(k^2 + m_{\vec{\sigma}}^2\right]\ , \\
\Omega_\pi &= N_f^2\,\frac{1}{2\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\ \ln\left[\beta^2(k^2 + m_\pi^2)\right]\ .
\end{aligned} \qquad (14)$$

The chiral condensate $\sigma(T, \mu)$ is determined by $\partial\Omega/\partial\sigma = 0$, which, after dividing it by $\lambda\sigma$, neglecting the multi-loop contributions, discarding the infinite vacuum contributions [10] and using $\langle\sigma'\rangle = 0$, leads to

$$\begin{aligned}
\sigma^2 &= f_\pi^2 - \frac{4N_f N_c g^2}{\lambda + \frac{\gamma}{N_f}}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{2\omega_F}\ n_F(\omega_F) \\
&\quad - \frac{N_f\lambda + 3\gamma}{N_f\lambda + \gamma}(N_f^2 - 1)\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{\omega_{\vec{\sigma}}}\ n_B(\omega_{\vec{\sigma}}) \\
&\quad - 3\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\ \frac{1}{\omega_{\sigma_0}}\ n_B(\omega_{\sigma_0}) - N_f^2\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\ \frac{1}{\omega_\pi}\ n_B(\omega_\pi)\ .
\end{aligned} \qquad (15)$$

From the results obtained for $N_f = 2$ [1], we conclude that (11) generalizes to

$$\begin{aligned} m_\pi &= 0 \, , \\ m^2_{\sigma_0} &= 2\left(\lambda + \frac{\gamma}{N_f}\right)\sigma^2 \, , \\ m^2_{\vec{\sigma}} &= \frac{2\gamma}{N_f}\sigma^2 \, , \\ m_F &= g\sigma \, , \end{aligned} \tag{16}$$

for $T, \mu \leq T_c, \mu_c$, where T_c and μ_c are the critical temperature and the critical chemical potential, respectively. We solve (15) numerically for $N_f = 3$, the results being depicted in Fig. 1. The results look qualitatively similar to the analogous results for $N_f = 2$ [1]. In particular, we find again that the phase transition is of the first order. However, the critical temperature for $N_f = 3$ is smaller than that for $N_f = 2$.

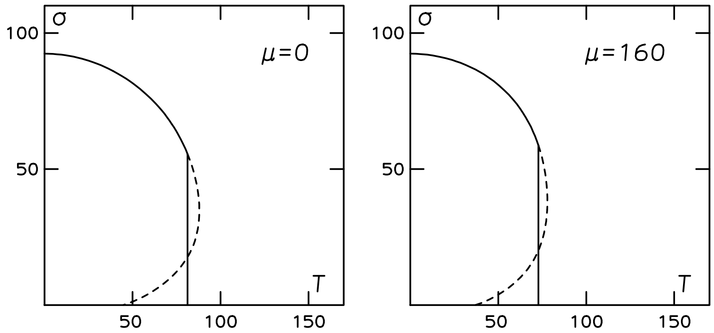

Fig. 1. Chiral condensate as a function of temperature at $\mu = 0$ and $\mu = 160$ MeV, for $N_f = 3$. The vertical line corresponds to the first-order phase transition. The dashed line corresponds to a physically unstable solution. σ and T are in MeV.

Finally, let us draw the conclusions. The results of Ref. 1 for $N_f = 2$ can be easily generalized to $N_f \geq 3$, providing that we choose a vacuum in which only $\langle \sigma_0 \rangle$ is different from zero. The first-order phase transition for $N_f = 3$ (with $\tau = 0$) occurs at a lower temperature than that for $N_f = 2$, which confirms the conclusion of Ref. 1 that the nuclear matter produced in heavy-ion collisions is close to or slightly above the chiral-phase-transition line. It is hard to say whether the prediction of the first-order transition is in agreement with present lattice simulations, because some lattice calculations predict a second-order transition (e.g. Ref. 11), while others predict a first-order one (e.g. Ref. 12).

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OBNAVLJANJE KIRALNE SIMETRIJE U LINEARNOM SIGMA MODELU

Proučava se kiralni fazni prijelaz u linearnom sigma modelu pri konačnoj temperaturi i barionskoj gustoći sa $N_f \geq 3$ kvarkovska okusa i N_c boja. Računi s jednom petljom za $N_f = 3$ predviđaju prijelaz prve vrste za $\mu = 0$ i $\mu \neq 0$. Svoje rezultate uspoređujemo s objavljenim rezultatima za $N_f = 2$.