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THE QUANTIZATION OF THE SOLAR-LIKE GRAVITATIONAL SYSTEMS

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Mean orbital distances r_n of planets from the Sun and of major satellites from the parent planets Jupiter, Saturn and Uranus are described by the square law $r_n = r_1 n^2$, where the values of n are consecutive integers, and r_1 is the mean orbital distance expected at $n = 1$ for a particular system. Terrestrial planets and Jovian planets are analysed as separate systems. Thus, five independent solar-like systems are considered. The basic assumption is that specific orbital angular momentum is "quantized". Consequently, all orbital parameters are also discrete. The number n relates to the law of orbital spacing. An additional discretization, related to r_1, i.e. to the scale of orbits, accounts for the detailed structure of planar gravitational systems. Consequently, it is also found that orbital velocity v_n multiplied by n is equal to the multiple of a fundamental velocity $v_0 \approx 24$ km s^{-1}, valid for all subsystems in the Solar System. This velocity is equal to one of the "velocity" increments of quantized redshifts of galaxies.

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1. Introduction

Recently, Agnese and Festa [1] published their approach in explaining discrete orbital spacing of planets in the Solar System. They used Bohr-Sommerfeld quantization rules and obtained the square law for orbital radii of planets in the form $a_n = a_1 n^2,\ n = 1, 2, 3 \dots$ All planets have been treated as one group. That assumption leads to many vacant orbits.

For example, Jupiter and Saturn occupy the orbits at $n = 11$ and $n = 15$, respectively, leaving three vacant orbits in between. Likewise, there are five vacant orbits between Saturn and Uranus. However, according to the current views [2], the planets are about as closely spaced as they could possibly be. Less massive planets are expected to be in more tightly packed orbits than the larger ones.

Recently, Oliveira Neto [3] used the square law in the form $r_{n,m} = r_0(n^2 + m^2)/2$, where n and m are integers. Only for Venus, Earth, Mars and Vesta m is not equal to n, while $n = m$ for all other planets, asteroid Camilla, Chiron and an unknown planet between Uranus and Neptune. Moreover, an average mass of all planets and asteroids equal to about 35 Earth masses is assumed in the calculation, which is not physically justified.

In our earlier work [4,5], we have shown that a square law could be applied to planetary orbital mean distances, as well as to those of major satellites of Jupiter, Saturn and Uranus. The leading assumption was that vacant orbits should be avoided. A radical change in treating the planetary orbits has been made by the separation of terrestrial planets from the Jovian ones. It means that terrestrial planets are considered as an independent system, enjoying the same status as the Jovian group of planets as well as the satellite system of Jupiter, Saturn and Uranus. The division of planets into two groups is justified by their different physical, chemical and dynamical properties [4,6,7]. From a cosmogonical point of view, an explanation could be the following: the centres of aggregation of future planets have been governed by the simple square law. After the accretion process, Jupiter has been formed in the orbit at $n = 2$, Saturn at $n = 3$, ending with Pluto at $n = 6$. The first Jovian protoplanet close to the Sun at $n = 1$, has never been formed due to the Sun's thermonuclear reactions. The high-melting-point materials have survived and accreted as the system of terrestrial planets, while the gaseous components have been dispersed due to the solar wind. Only beyond the "temperature limit" of about 200 K, which corresponds to about $5 \cdot 10^{11}$ m, could the giant Jovian planets exist [4].

The division of planets into two groups appeared also in solving the modified Schrödinger radial equation of the hydrogen-like atom introducing, of course, the gravitational potential [8] and coefficient of diffusion of Brownian motion which characterizes the effect of chaos on large time scales [9a,10]. From a dynamical point of view, the five systems: terrestrial planets, Jovian planets, and satellites of Jupiter, Saturn and Uranus are to a considerable degree adiabatic. Therefore, the relevant equations in the present model include characteristic parameters of the particular system, but they also have a necessary physical generality and consistency. However, many authors [7,11,12] have prefered to treat the spacing of all planets with a single formula, like the Titius-Bode law or its numerous modifications. The authors of this work consider the square law, like that discovered by Bohr in his planetary model of the hydrogen atom, more favourable for an analysis of the planar gravitational systems. Moreover, it has been proposed [13] that the square law of orbital spacing, could be termed the fourth Kepler's law, in the honour of Kepler who searched for a rule of planetary spacing about four centuries ago.

An application of the square law to the extra-solar planetary systems will certainly be examined in the near future. Recently, first attempts [5,10] were made for the three planets of pulsar PSR B 1257+12.

2. The model

A discrete distribution of planetary orbits may be obtained by the "quantization" of angular momentum J_n. Let an orbiting mass be denoted by m_n, and mass of the central body by M. Then, using Newton's equation of motion for circular orbits, angular momentum (supposing that $m_n << M$) is given by

$$J_n = m_n v_n r_n = m_n \sqrt{GMr_n}, \tag{1}$$

where G is the gravitational constant, r_n is the radius of the n-th orbit and v_n is the orbital velocity. We assume that angular momentum is "quantized",

$$m_n \sqrt{GMr_n} = n\frac{H}{2\pi}, \tag{2}$$

where H may be treated as an effective " Planck's gravitational constant", depending on the particular system and even on the particular orbiting body. Equation (2) is not very useful. What one can do is to divide $H/2\pi$ by the mass of the orbiting body to obtain the "specific Planck's constant" $H' = H/(2\pi m_n)$ which yields for the orbital radius

$$r_n = \frac{n^2 H'^2}{GM}. \tag{3}$$

H' is also system dependent, but the quantity H'/M is of the same order of magnitude for all systems (see Table 1, and also Ref. 4). Variability of H'/M is described by a dimensionless factor f multiplied by a universal constant A, i.e., $H/(2\pi m_n M) = H'/M = fA$. Then, Eq. (3) takes the form

$$r_n = \frac{1}{G}(fA)^2 M n^2. \tag{4}$$

We have shown [4] that by comparing electrostatic and gravitational forces, as one possible approach, the constant A may be defined by the fundamental physical constants as follows:

$$A = 2\pi\frac{G}{\alpha c} = 1.9157 \cdot 10^{-16}\ \mathrm{m}^2\ \mathrm{kg}^{-1}\ \mathrm{s}^{-1}, \tag{5}$$

where $\alpha = 2\pi e^2/(4\pi\epsilon_0 hc)$ is the fine-structure constant, e the charge of an electron, ϵ_0 the permitivity of vacuum, h the Planck constant and c the velocity of light. The dimension of the constant A is that of angular momentum per square mass, and, in accordance with Eq. (5), the simple proportionality between A and the Planck constant per square Planck's mass $m_P = (hc/(2\pi G))^{1/2} = 2.177 \cdot 10^{-8}$ kg [9b] is given by

$$A = \frac{h}{\alpha m_P^2}, \qquad \text{or} \qquad A = \frac{h}{m_0^2}, \tag{6}$$

where $m_0^2 = \alpha m_P^2$. A constant analogous to A has been defined as $p = 0.8 \cdot 10^{-16}$ m^2 kg^{-1} s^{-1} by Wesson [14] in searching for a clue to a unification of gravitation and particle physics. Such a constant appeared also in Ref. 1 with the value $2.35 \cdot 10^{-16}$ $\mathrm{m^2\ kg^{-1}\ s^{-1}}$. Slightly different values of the same constant are due to different initial assumptions.

TABLE 1. Mean values of constants r_1, H'/M and f, with the assigned values of integers n, for planetary and satellite systems.

System	r_1 (m)	n	H'/M ($\mathrm{m^2s^{-1}kg^{-1}}$)	f
Terrestrial planets	$(0.639 \pm 0.016)10^{10}$	3,4,5,6, 8	$(0.462 \pm 0.006)10^{-15}$	2.41 ± 0.03
Jovian planets	$(1.751 \pm 0.044)10^{11}$	2,3,4,5, 6	$(2.418 \pm 0.030)10^{-15}$	12.61 ± 0.16
Jupiter's satellites	$(4.579 \pm 0.180)10^{7}$	2,3,4,5, 6	$(1.268 \pm 0.025)10^{-15}$	6.62 ± 0.13
Saturn's satellites	$(0.390 \pm 0.012)10^{7}$	6,7,8,9, 10,11	$(0.676 \pm 0.010)10^{-15}$	3.53 ± 0.05
Uranus' satellites	$(0.843 \pm 0.018)10^{7}$	3,4,5,6, 7,8	$(2.542 \pm 0.027)10^{-15}$	13.27 ± 0.28

Using Eqs. (4) and (5), some important parameters of the solar subsystems, the orbital radii $r_n = r_1 n^2$, specific angular momenta J_n/m_n, orbital periods T_n and velocities v_n are given by

$$r_n = \left(\frac{2\pi f}{\alpha c}\right)^2 GMn^2, \tag{7}$$

$$\frac{J_n}{m_n} = \left(\frac{2\pi f}{\alpha c}\right) GMn \qquad (8),$$

$$T_n = 2\pi \left(\frac{2\pi f}{\alpha c}\right)^3 GMn^3 \qquad (9),$$

$$v_n = \frac{1}{2\pi f}\frac{\alpha c}{n}. \tag{10}$$

In Eq. (10), $\alpha c/n = v_{nH}$ is the orbital velocity of an electron at the n-th orbit in the Bohr's model of the hydrogen atom, and the term $1/(2\pi f)$ is a gravitational correction factor. This term is system dependent and it demonstrates that gravitational systems are less regular than analogous electrodynamical systems.

3. *Results and discussion*

Distributions of specific angular momenta of planets and major satellites according to the linear relationship (Eq. (8)) are illustrated in Fig. 1.

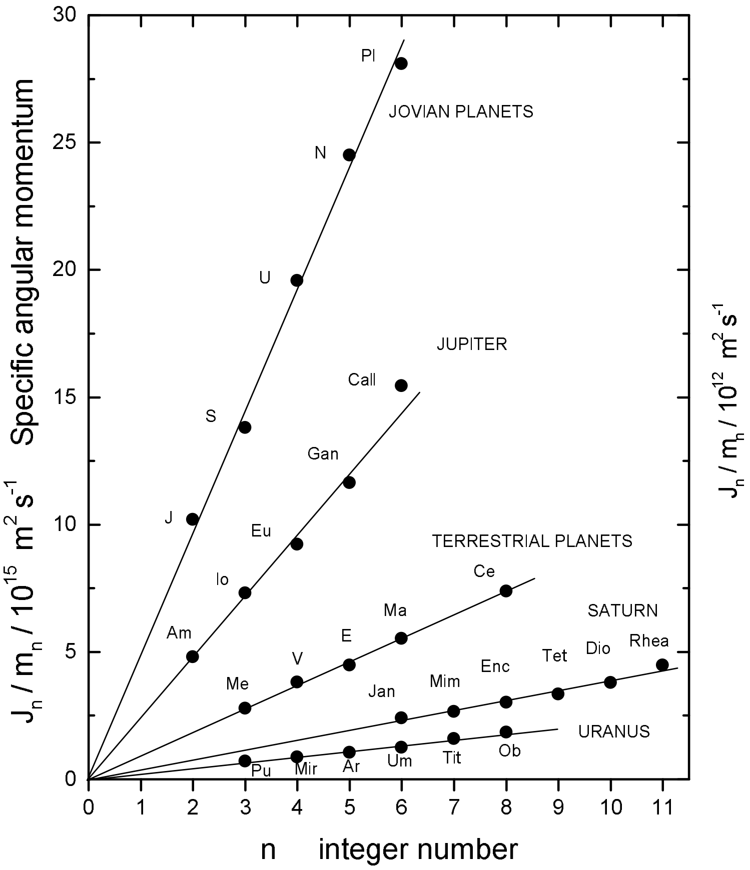

Fig. 1 Specific angular momentum $J_n/m_n = \sqrt{GMr_n}$ versus the integer number n for Jovian and terrestrial planets (left scale) and for the major satellites of Jupiter, Saturn and Uranus (right scale).

Discrete values of J_n/m_n are obtained from Eq. (1) using the observed values of semi-major axes as the mean distances of planets from the Sun, or of satellites from the parent planet, which are taken as the orbital radii r_n of approximate circular orbits. This introduces small errors of r_n [4], and of J_n/m_n for Mercury and Pluto, due to the eccentricities of their orbits of 0.206 and 0.255, respectively [15]. The approximation of circular orbits is very good for other planets and all major satellites. The integer numbers n are unambiguously determined by the requirement of Eq. (8) that angular momenta are zero at $n = 0$, resulting in the straight lines shown in Fig. 1, with no intercepts, as the best fits to the deduced values of J_n/m_n. The left scale corresponds to Jovian and terrestrial planets, while the right scale is valid for major satellites of Jupiter, Saturn and Uranus. We have also included in our calculations the satellites Amalthea, Janus and Puck (the largest of the small ones), which are near the Roche limit of the parent planets Jupiter, Saturn and Uranus, respectively, and also the largest asteroid Ceres. Therefore, the values of r_1 in the square law $r_n = r_1 n^2$ for spacing of planetary orbits in accordance with Eq. (7), and also of H'/M and f, which are listed in Table 1, differ slightly from the values given in our earlier work [4]. Note that the orbit of the asteroid Ceres is at $n = 8$, which is nearly the center of the Main Belt, whose extention is from $n = 7$ to $n = 9$.

There is one exception in treating the spacing of major satellites. Titan, the largest satellite of Saturn, is not included in the system of smaller satellites from Janus to Rhea. Titan would have the orbit at $n = 19$ if it were a member of that system. Seven vacant orbits between Rhea and Titan suggest that Titan could be a member of a more extensive system, similarly to Jupiter in the Jovian group of planets in relation to the terrestrial planets. Titan and small satellites Hyperion and Japetus do not form a complete system.

Note that asteroids (except for the largest, Ceres), comets, planetary rings and outer small satellites of planets can not be treated by Eqs. (7-10) because, due to their small masses, a variety of other physical processes (scattering, capture, impacts, planetary perturbations) prevail over the simple law. Moreover, it was recently shown in modeling the massive extrasolar planets, that orbital evolution and significant migration of planets could take place, due to the interaction of a planet with circumstellar disk, with the parent spinning star and also due to the Roche lobe overflow [16]. A planet may move very far from its initial position of formation accompanied also with the loss of mass. However, under certain conditions, planets maintain their position of formation. One may suppose that initial positions are governed by the square law according to the ”quantum-mechanical laws”, but possible later evolution might be subjected to numerous “effects of classical physics”.

We have tried to correlate the factor f with the ratio of the total mass (m_n of orbiting bodies to the mass M of the central body [5], more precisely, of f with $(\sum m_n/M)^{1/3}$. The values of f for terrestrial planets, Jovian planets and satellites of Jupiter fit very well a straight line, but there are strong deviations of f for satellites of Saturn, and particularly for those of Uranus. Note that the planes of planetary orbits are close to the ecliptic (except those of Mercury and Pluto) which is also valid for satellites of Jupiter, due to the small inclination of Jupiter's spin axis. However, the satellites of Saturn have an inclination of $27°$ and those of Uranus $98°$. Their satellites have supposedly been formed in the equatorial planes after the protoplanets, within the planetary envelopes, and obtained an additional angular momentum of yet unknown origin. We believe that the deviation of the factor f from the introduced correlation [5] has the same cause as the change of inclination.

In our later investigation, we have found that reciprocal values of the factor f take discrete values that may be described by another integer number k, i.e.,

$$f^{-1} = (0.06753 \pm 0.00085)k + (0.0115 \pm 0.0029), \tag{11}$$

as may be seen in Fig. 2. Therefore, Eq. (10) may be written in the form

$$nv_n = v_1 \approx [(23.5 \pm 0.3)k + (4.0 \pm 1.0)]\mathrm{km\ s^{-1}}. \tag{12}$$

The product of nv_n, i.e. the orbital speed v_1 at $n = 1$ for a particular system, vs. n is shown in Fig. 3. The values of n are taken from Table 1, and the mean velocities from observed semimajor axes as $v_n = (GM/r_n)^{1/2}$ (see

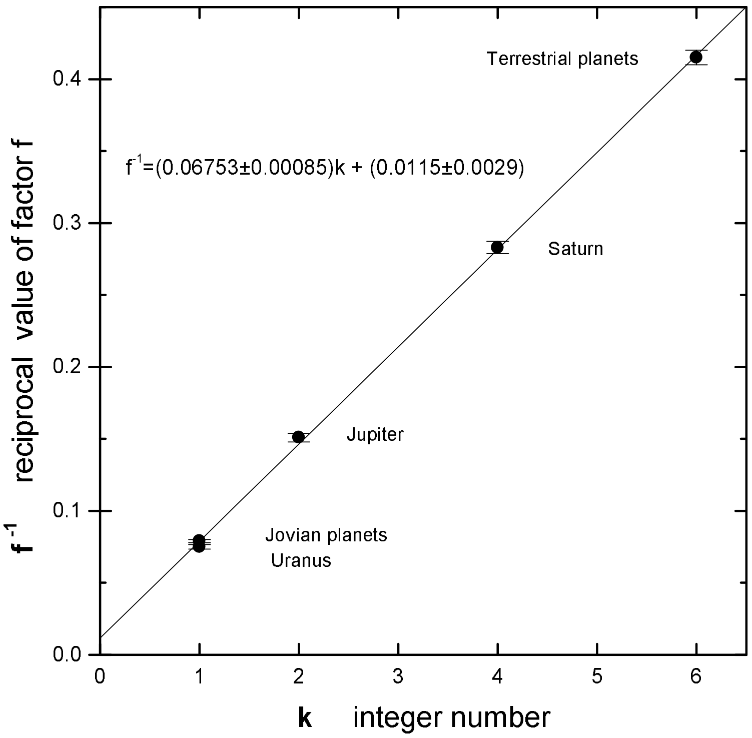

Fig. 2. Correlation of the reciprocal value of the factor f with integer number k.

Horizontal lines represent "velocity levels" with spacing defined by $v_0 = 23.5$ km s^{-1} (Eq. (12)). The integer number k is related to the scale of orbits in a system. It means that a given system can have a series of discrete possible orbital distributions. That is hardly understandable from the standpoints of classical physics, because one can only expect a continuous change of orbital spacing. For example, Uranian satellites are characterized by $k = 1$. Neglecting the value of f^{-1} at $k = 0$ in Eq. (11), the orbital radii are approximately described by $r_n = const \cdot Mn^2/k^2$. If $k = 2$, the orbits would be contracted by the factor four, i.e. contraction of orbits occurs in jumps. Consequently, reduced orbital radii r_n/M become

$$\frac{r_n}{M} = \frac{G}{v_0^2}\frac{n^2}{k^2} = (1.07 \pm 0.06) \cdot 10^{-19}\,\frac{n^2}{k^2}, \tag{13}$$

where G/v_0^2 may be called a characteristic length with a dimension mkg^{-1}. The value of v_0 in Eq. (13), equal to (25.0 ± 0.7) km s^{-1} was obtained from the fit of f^{-1} vs. k with zero intercept at $k = 0$, and neglecting a constant term of velocity v_1 at $k = 0$ (i.e., 4.0 km s^{-1} in Eq. (12)). That causes a larger error in the calculation of r_n, v_1 and of other quantities, but the formulae are simpler in illustrating the main

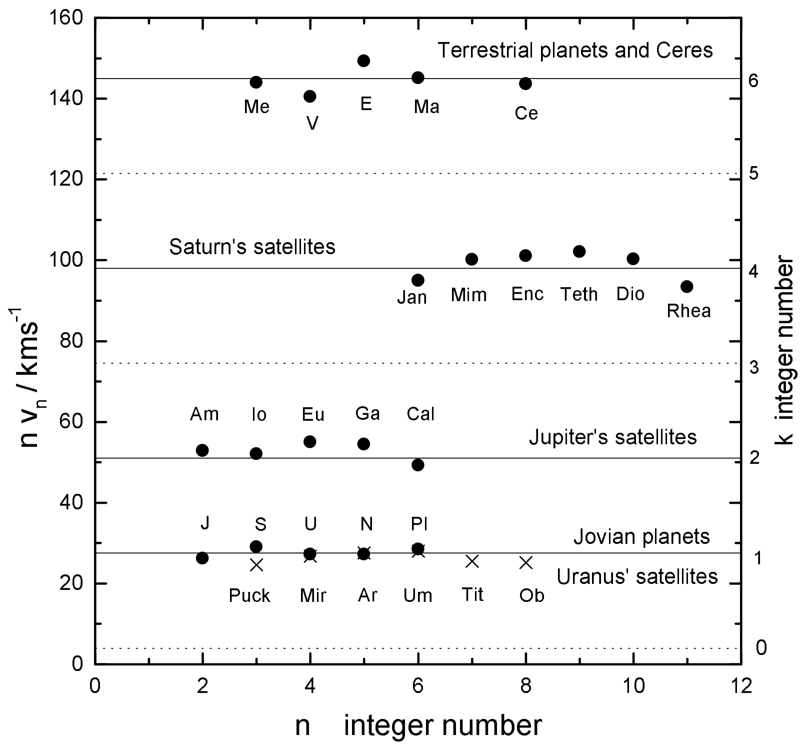

Fig. 3. The product nv_n of mean orbital velocity $v_n = \sqrt{GM/r_n}$ and integer number n versus n for Jovian and terrestrial planets and for the major satellites of Jupiter, Saturn and Uranus. Integer number k (right scale) is related to the scaling of orbits. The "velocity levels" are given by Eq. (12).

The orbital integers n and k determine the details of possible discrete gravitational structures.

The value of $v_0 \approx 24$ km s^{-1} has been found as one of increments of the intrinsic galactic redshifts derived from their "quantized" values [17-21]. One may suspect that v_0 is important not only for the Solar System, but that it has a deeper physical meaning to be revealed.

Equation (13) may be rewritten in another important, symmetrical form

$$\frac{r_n}{M} v_0^2 k^2 = \frac{G}{c^2} c^2 n^2. \tag{14}$$

The term G/c^2 is equal to the ratio of the Planck's length $L_P = (hG/(2\pi c^3))^{1/2}$ and Planck's mass $m_P = (hc/(2\pi G))^{1/2}$, i.e., $G/c^2 = L_P/m_P$. Hence, Eq. (14) takes a form

$$\frac{r_n}{M} (kv_0)^2 = \frac{L_P}{m_P} (nc)^2. \tag{15}$$

Equation (15) gives a remarkable connection between macroscopic and microscopic parameters of gravitational systems.

Consider again the initial assumption in our model. The discretization of angular momenta, using the approximation of circular orbits, is given by Eq. (2), i.e., $m_n v_n r_n = nH/2\pi$. The present model permits to write a proper "quantum condition" in accordance with Bohr as

$$m_n v_n r_n \left(\frac{M m_n}{m_0^2} 2\pi f\right)^{-1} = n\frac{h}{2\pi}. \tag{16}$$

An approach to prove Eq. (16), using the theory of similarity, is given in Appendix. Equation (16) can be interpreted as follows: angular momentum of an orbiting body in a planar gravitational system is proportional to the mass M of the central body and to the mass m_n of the orbiting body. Therefore, angular momentum per square mass is of special importance. Further multiplication by $m_0^2 = \alpha m_P^2$ scales a gravitational macroscopic system to the microscopic (atomic) one. However, the ratio Mm_n/m_0^2 must be multiplied by the factor $2\pi f$, which has to be determined from observational data. Dynamic properties of gravitational systems reach, in the limit, the electrodynamical ones. If the quantities $r_n = GM/v_n^2$ and $m_0^2 = h/A$ are introduced in Eq. (16), one easily obtains $v_n = (2\pi f)^{-1}\alpha c/n$ for the velocity at the n-th orbit, in accordance with Eq. (10). For $2\pi f = 1$, the orbital velocity distribution of the electron in Bohr's hydrogen atom is obtained. It has already been shown that $nv_n = v_1 = \alpha c/(2\pi f) = kv_0$ (see Fig. 3). For $k = 1$, one obtains $v_0 = 25.0$ km s^{-1}, and consequently, from $v_0 = \alpha c/(2\pi f_0)$ follows that the maximum value of $2\pi f$ is $2\pi f_0 = 87.6 \pm 2.5$. Orbital radii are then simply given by $r_n = GM/v_n^2 = (2\pi f_0/(\alpha c))^2 GMn^2/k^2$, which is just Eq. (13). From Eq. (16), an effective "Planck's gravitational constant" appears to be $H = (2\pi f M m_n/m_0^2)h$. Then, the Schrödinger's radial wave equation for a gravitational system generates the first orbital radius r_1 in agreement with Eq. (7), as it is shown in Appendix.

The present model describes the structures of planar gravitational systems. It includes three parameters: two integer numbers, n and k, and a factor f_0 or velocity v_0. Eqs. (7-10) may be written in an approximate form as

$$r_n = \frac{1}{v_0^2} GM \frac{n^2}{k^2}, \tag{17}$$

$$\frac{J_n}{m_n} = \frac{1}{v_0} GM \frac{n}{k}, \tag{18}$$

$$T_n = 2\pi \frac{1}{v_0^3} GM \frac{n^3}{k^3}, \tag{19}$$

$$v_n = v_0 \frac{k}{n}. \tag{20}$$

According to Eq. (12), $nv_n \approx (23.5k + 4.0)$ km s^{-1}. Therefore, Eq. (20) deviates from the best fit (Eq. (12)) by the factor $(1 - 25k/(23.5k + 4.0))$, i.e. by about 9% if $k = 1$, and by about -3% if $k = 6$, while observational mean values of $nv_n = n(GM/r_n)^{1/2}$ deviate from the best fit (Eq. (12)) less than 2% on the average.

One may criticize the use of many parameters in the model. However, they seem to be necessary, because n is related to the principal spacing of orbits, k takes care of the packing of orbits, while v_0 (or f_0) characterizes several subsystems within a given system (like our own Solar System). One should not be surprised if in another extra-solar system, the quantity v_0 would take a different value compared with the Solar System. It could possibly be 72, 36, 24, or 18 km s^{-1}, as obtained in an analysis of the quantized redshifts of the galaxies [17–20]. For example, the pulsar PSR B 1257+12 has three planets in orbits for n equal to 5, 7 and 8 [5,10]. From the observational data, one obtains $nv_n = 410$ km s^{-1}, which gives $k = 17$ for $v_0 = 24$ km s^{-1}. However, if one assumes $v_0 = 37.3$ km s^{-1} (in accordance with Ref. 21, where the interval for redshift periodicity is 37.2 to 37.7 km s$^{-1)}$, then k will be equal to 11. Hopefully, the future investigation of other planetary systems will confirm the ideas proposed in the present model.

4. *Conclusion*

The basis of the square law for the spacing of orbits of planets and of major satellites is the discretization of angular momenta, similarly as in the old Bohr's theory of the hydrogen atom. However, the angular momentum of an orbiting body has to be reduced by the mass of orbiting body and also by the mass of the central body. Moreover, the product of these two masses must also be reduced by square of Planck's mass multiplied by the fine-structure constant α, in order to scale the macroscopic gravitational system to the microscopic level, where the Planck's reduced constant $\alpha = h/2\pi$ represents a quantum of angular momentum. As a result of such an approach, two "quantum numbers" appear, the first one n for describing the law of orbital spacing and the second one k for the "packing" of the orbits. One further parameter is necessary, that is equal for all systems within the Solar System. It is the characteristic length $G/v_0^2 = (1.07 \pm 0.06)10^{-19}$ m kg^{-1}. But equally well, the third parameter may be a universal velocity $v_0 \approx 24$ km s^{-1}. The three parameters and the mass of the central body (see Eqs. (17-20)) define possible the discrete structures of a planar gravitational system within the approximation of the circular orbits.

Velocity v_0 is equal to the velocity increments of the quantized redshifts of galaxies. A great puzzle is how the planetary orbital velocities can obey the same quantization periods as the intrinsic redshifts of the galaxies.

It is known that some researches do not believe that "quantum phenomena" play any role, both in the formation and in the evolution of the Solar System. They rather suppose that many macroscopic effects have had a predominant influence on planetary spacing. However, in our opinion, the derived results shown in Figs. 1 to 3 strongly suggest the necessity for a certain "quantum mechanical" treatment. As the first approach, the model analogous to the simplest one of the "old quantum mechanics" has been elaborated in the present work. Of course, further observational and theoretical investigations are necessary for the development of more sophisticated models.

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Appendix

The similarity between the gravitational and Coulomb force between two particles of mass m_0 and charge e is well known. Moreover, one can imagine that these two forces become identical for adequately chosen mass m_0. From $Gm_0^2/r^2 = e^2/(4\pi\epsilon_0 r^2)$, it follows that $m_0 = (e^2/(4\pi\epsilon_0 G))^{1/2} = (\alpha ch/(2\pi G))^{1/2}$, independently of the mutual distance of particles. The mass m_0 is related to the Planck's mass m_P by $m_0 = \alpha^{1/2} m_P = 1.85910^{-9}$ kg. It is reasonable to assume that for such a micro-gravitational system, a quantization of angular momentum of the orbiting body should be the same as the one postulated by Bohr for the electrodynamical system, i.e.,

$$m_0 v_{n0} r_{n0} = n \frac{h}{2\pi}. \qquad (A1)$$

For a real macro-gravitational system an analogous discretization could be

$$m_n v_n r_n = n \frac{H}{2\pi}. \qquad (A2)$$

To reach a complete similarity between the reference micro-model and a real planetary or satellite system, analogous quantities must be in a constant ratio. These ratios, the so-called similarity constants, such as $N_m = m_n/m_0$, $N_v = v_n/vn_0$ and $N_r = r_n/rn_0$, must be in definite mutual relationships, which can be generally determined from analogous equations [22]. Thus, Eq. (A1) will transform into Eq. (A2) only with the correlation

$$N_m N_v N_r = N_h = \frac{H}{h}, \qquad (A3)$$

which is an indicator of similarity, satisfied for every orbit and for any value of n. To determine H, an additional indicator of similarity must be taken into account, which follows from analogous correlations for the forces corresponding to the micro-model and to a system of a body (of mass m_n) orbiting the central one (of mass M):

$$v_{n0}^2 r_{n0} = Gm_0, \qquad (A4)$$

$$v_n^2 r_n = GM, \qquad \text{and} \qquad (A5)$$

$$N_v^2 N_r = \frac{M}{m_0}. \qquad (A6)$$

Introducing the second indicator of similarity (A6) into the first one (A3), one obtains $H/h = Mm_n/m_0^2 N_v$. Further, from Eqs. (A1) and (A4) for $m_0 = \alpha^{1/2} m_P$, it follows

$v_{n0} = \alpha c/n$, and according to Eq. (10), $N_v = v_n/v_{n0} = (2\pi f)^{-1}$. Thus, the effective "Planck's gravitational constant" H is given by

$$H = h(2\pi f \frac{M m_n}{m_0^2}), \tag{A7}$$

where the factor f (see Table 1), determined from astronomical data, is included.

Finally, by introducing Eq. (A7) into Eq. (A2), the scaled "quantum condition" presented by Eq. (16) is proved.

Consequently, Eq. (A7) should be used, e.g., to define a macroscopic "de Broglie wavelength" $\lambda_n = H/m_n v_n$. Introducing v_n from Eq. (10), one obtains $\lambda_n = 2\pi r_n/n$, where r_n is given by Eq. (7). This is an expected result in the present model. λ_n may be transformed into a form dependent on n and k as $\lambda_n = (2\pi/v_0^2)GMn/k^2$ by using Eq. (17). One may also write $\lambda_n = \lambda_1 n$, which is an equivalent simple form of the square law $r_n = r_1 n^2$.

Equation (A7) allows the use of the Schrödinger's radial wave equation [8] to obtain the orbital spacing. If the gravitational potential $V(r) = -GMm/r$ and effective "Planck's gravitational constant" H are introduced into the radial equation, it takes the form

$$\frac{\mathrm{d}R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{8\pi^2 m^2 E'}{H^2}R + \frac{2}{r}\frac{4\pi^2 GMm^2}{H^2}R - \frac{l(l+1)}{r^2}R = 0, \tag{A8}$$

where $E' = E/m$ is the energy per unit mass of the orbiting body, $R(r)$ is the radial wave function and l is the angular quantum number. From the fourth term, "the first Bohr's radius" is

$$r_1 = \frac{H^2}{4\pi^2 GMm^2}. \tag{A9}$$

Introducing H defined by Eq. (A7), with $m_n = m$, one obtains

$$r_1 = \left(\frac{2\pi f}{\alpha c}\right)^2 GM, \tag{A10}$$

which is in agreement with Eq. (7) for $n = 1$. If the angular quantum number is limited only to the values $l = n - 1$, then the probability maxima of the mass distribution will be at positions given by $r_n = r_1 n^2$. Such an approximation has been recently used by Nottale et al. [23]. If all wave functions up to $n = 10$, with all possible values of l are used [8], then the positions of probability maxima slightly deviate from the square law. However, it was already pointed out that the simple approach, related to the old quantum theory is more appropriate for an understanding of gravitational phenomena [24]. Therefore, the complete understanding of the rather formal application of the Schrödinger's equation to the Solar System needs further research.

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KVANTIZACIJA GRAVITACIJSKIH SUSTAVA SLIČNIH PLANETARNOM SUSTAVU SUNCA

Srednje orbitalne udaljenosti planeta od Sunca i glavnih satelita od planeta Jupitra, Saturna i Urana opisane su kvadratnim zakonom $r_n = r_1 n^2$, gdje je n sukcesivno rastući cijeli broj, a r_1 je srednja orbitalna udaljenost za $n = 1$. Terestrički i jovijanski planeti razmatrani su kao nezavisni sustavi, pa zajedno sa satelitima triju spomenutih planeta daju pet sličnih planarnih gravitacijskih sustava. Polazna pretpostavka je "kvantiziranost" specifičnog orbitalnog momenta impulsa. Sukladno tome i svi ostali dinamički parametri sustava poprimaju diskretne vrijednosti. Broj n određuje zakonitost porasta orbitalnih udaljenosti planeta ili satelita, no pored toga uočeno je da i veličina r_1, nezavisno od n, poprima diskretne vrijednosti. To znači da pojedini sustav može imati niz struktura različite gustoće orbita. Jedna od posljedica toga je da produkt orbitalnih brzina v_n i pripadnog broja n postaje konstantan za dani sustav i ujedno je višekratnik osnovne brzine $v_0 \approx 24$ km s^{-1}. Ova brzina jednaka je jednoj od "brzina" izvedenih iz kvantiziranih crvenih pomaka galaksija, pa ona možda ima i neko dublje fizičko kozmološko značenje.