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LETTER TO THE EDITOR

ON THE EXISTENCE OF ADDITIONAL SOLUTIONS OF THE EQUATIONS IN THE $(1/2,0)\oplus(0,1/2)$ REPRESENTATION SPACE

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We analyze dispersion relations of the equations recently proposed by Ahluwalia for describing the neutrino. Equations for type-II spinors are deduced on the basis of the Wigner rules for left- and right- 2-spinors and the Ryder-Burgard relation. It is shown that the equations contain acausal solutions which are similar to those of the Dirac-like second-order equation. The latter is obtained in a similar way, provided that we do not apply any constraints in the calculation.

Recently, Ahluwalia proposed a new wave equation for describing the self/anti-self charge conjugate states $\lambda^{S,A}(p^\mu)$ of any spin [1]:

$$\mathcal{D}\lambda(p^\mu) = \tag{1}$$

$$\begin{pmatrix} -\mathbf{1} & \zeta_\lambda \exp(\vec{J}\cdot\vec{\varphi})\,\Theta_{[j]}\,\Xi_{[j]}\exp(\vec{J}\cdot\vec{\varphi}) \\ \zeta_\lambda \exp(-\vec{J}\cdot\vec{\varphi})\,\Xi^{-1}_{[j]}\,\Theta_{[j]}\exp(-\vec{J}\cdot\vec{\varphi}) & -\mathbf{1} \end{pmatrix} \lambda(p^\mu)$$

$$= 0\,.$$

Analogous equations for $\rho^{S,A}(p^\mu)$ bispinors have been derived in Ref. 2d. In the $j = 1/2$ case, spin matrices $\vec{J}$ are chosen to be the Pauli matrices $\vec{\sigma}/2$, and in the $j = 1$ case, the Barut-Muzinich-Williams matrices; $\vec{\phi}$ are the parameters of the Lorentz boost. The notation is the same as that of Refs. 1 and 2. $\Theta_{[j]}$ is the Wigner's operator defined by

$$(\Theta_{[j]})_{\sigma,\sigma'} = (-1)^{j+\sigma}\delta_{\sigma',-\sigma},$$

with σ and σ' as eigenvalues of $\vec{J}$. $\Xi_{[j]}$ is a $(2j+1)\times(2j+1)$ matrix which connects the "$2j+1$ spinor" and its complex conjugate such that

$$[\phi_L^\sigma(\mathring{p}^\mu)]^* = \Xi_{[j]}\phi_L^\sigma(\mathring{p}^\mu).$$

And ζ_λ are phase factors which are fixed by the conditions of self/anti-self charge conjugacy imposed on the type-II spinors λ.

While formally the $j = 1/2$ equation "may be put in the form

$$(\Gamma^{\mu\nu}p_\mu p_\nu + m\Gamma^\mu p_\mu - 2m^2\mathbf{1})\lambda(p^\mu) = 0$$

. . . , it turns out that $\Gamma^{\mu\nu}$ and Γ^μ do not transform as Poincaré tensors." Other forms of neutrino equations have been presented in Refs. 3, 2 and 4, and gauge interactions have been introduced there.[1] These constructs give alternative insights into the neutrino dynamics, which is different from that based on the commonly-used Weyl massless equations. Indications that neutrino may not be a Dirac particle and may have different dynamical features have appeared in analyses of the present experimental situation [5]. Earlier considerations of this problem can be found in Refs. 6–9.

Both Eq. (1) and the equations of Refs. 2, 4 and 10 have been obtained by using different forms of the Ryder-Burgard relation (Refs. 11, 12, 1, 2, 4 and 10) which relates zero-momentum $(0, j)$ left and $(j, 0)$ right spinors, and the Wigner rules for their transformations to the frame with the momentum $\vec{p}$. The Dirac equation may also be obtained in such a way (footnote # 1 of Ref. 1). The detailed discussion of this techniques can be found in Ref. 13. It was claimed in Ref. 1 that $\lambda^S(p^\mu)$ spinors correspond to "*positive* energy solutions, . . . , [while] $\lambda^A(p^\mu)$ are the *negative* energy solutions". We, in fact, used this interpretation in Ref. 2. Let us now check by straightforward calculations, which dispersion relations has Eq. (1) in the case of $j = 1/2$. Rewriting it to the form (31) of Ref. 1 yields the equation of the second order in p_0, and the matrix at the left side has the dimension four. So, one should have eight solutions. The system for analytical calculation MATEMATICA 2.2 enabled us to find the determinant of the matrix $\mathcal{D}$:

$$\mathrm{Det}\,[\mathcal{D}] = \left(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2\right)^2 \frac{(p_0^2 - p_1^2 - p_2^2 - p_3^2 + 3m^2 + 4mp_0)^2}{16m^4(p_0+m)^4}. \qquad (2)$$

As a result of equating the determinant to zero we deduce that, indeed, Eq. (1) has eight solutions in total with

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2}, \qquad (3)$$

[1]The question of equivalence of these equations still deserves further elaboration and this paper presents a certain part of this analysis.

each two times; and with the acausal dispersion relations:

$$p_0 = -2m \mp \sqrt{\vec{p}^{\,2} + m^2}, \tag{4}$$

each two times.

We come across the same situation when deriving the Dirac equation by the Ryder-Burgard-Ahluwalia technique, provided that we do *not* apply the constraint $p_0^2 - \vec{p}^{\,2} = m^2$ from the beginning. Indeed, one has

$$\Lambda_R(p^\mu \leftarrow \mathring{p}^\mu)\Lambda_L^{-1}(p^\mu \leftarrow \mathring{p}^\mu) = \frac{p_0^2 + 2mp_0 + \vec{p}^{\,2} + m^2 + 2(p_0+m)(\vec{\sigma}\cdot\vec{p})}{2m(p_0+m)}, \tag{5a}$$

$$\Lambda_L(p^\mu \leftarrow \mathring{p}^\mu)\Lambda_R^{-1}(p^\mu \leftarrow \mathring{p}^\mu) = \frac{p_0^2 + 2mp_0 + \vec{p}^{\,2} + m^2 - 2(p_0+m)(\vec{\sigma}\cdot\vec{p})}{2m(p_0+m)}. \tag{5b}$$

Thus, the second-order momentum-representation "Dirac" equation can be written:

$$\frac{1}{2m(p_0+m)}\left[(\gamma^\mu p_\mu \mp m)\gamma^0 + 2m\right](\gamma^\nu p_\nu \mp m)\Psi_\pm(p^\mu) = 0, \tag{6}$$

or

$$\frac{1}{2m(p_0+m)}(\gamma^\nu p_\nu \mp m)\left[\gamma^0(\gamma^\mu p_\mu \mp m) + 2m\right]\Psi_\pm(p^\mu) = 0. \tag{7}$$

The corresponding coordinate representation of these equations ($m \neq 0$ and $p_0 \neq -m$) is

$$\left[(i\gamma^\mu\partial_\mu - m)\gamma^0 + 2\wp_{u,v}m\right](i\gamma^\nu\partial_\nu - m)\Psi(x^\mu) = 0, \tag{8}$$

or

$$(i\gamma^\mu\partial_\mu - m)\left[\gamma^0(i\gamma^\mu\partial_\mu - m) + 2\wp_{u,v}m\right]\Psi(x^\mu) = 0, \tag{9}$$

where $\wp_{u,v} = \pm 1$ depending on what solutions, with either positive or negative energies, are considered.

Can Eq. (1) be put in a more convenient form? The eight-component form, we proposed recently (Ref. 2d, Eqs. (17,18)), does not have acausal solutions. In the process of its derivation, we assumed certain relations[2] between $\lambda^{S,A}(p^\mu)$ and $\rho^{S,A}(p^\mu)$. In the present article, we are not going to apply them. Following the procedure of deriving Eqs. (6,7) one can arrive at a rather complicated equation:

$$\frac{1}{4m(p_0+m)}\left\{(\gamma^\mu p_\mu + m\gamma^0)\left[\mathcal{S}(\gamma^\nu p_\nu + m\gamma^0) - 2m\gamma^0\right] + \right.$$
$$\left. + \quad \left[(\gamma^\mu p_\mu + m\gamma^0)\mathcal{S} - 2m\gamma^0\right](\gamma^\nu p_\nu + m\gamma^0)\right\}\lambda^{S,A}(p^\mu) = 0, \tag{10}$$

[2] See, e.g., formulas (48) of Ref. 1.

where

$$\mathcal{S} = \begin{pmatrix} 0 & \zeta_\lambda \Theta \Xi \\ \zeta_\lambda \Xi^{-1} \Theta & 0 \end{pmatrix}. \tag{11}$$

But, as mentioned in Ref. 1, one may consider that ϕ in the generalized Ryder-Burgard relation (see Eq. (27) of Ref. 1 or Eq. (38) in Ref. 2c) is the azimuthal angle *associated with* $\vec{p}$, the 3-momentum of a particle. In this case one can find commutation relations between $\hat{p} \equiv \gamma^\mu p_\mu$, matrices γ^5, γ^0 and $\mathcal{S}$:

$$[\hat{p}, \mathcal{S}]_- = 0, \quad [\gamma^0, \mathcal{S}]_- = 0, \quad \left[\gamma^5, \mathcal{S}\right]_+ = 0, \tag{12}$$

and

$$\mathcal{S} \lambda^{S,A}(p^\mu) = \lambda^{S,A}(p^\mu), \tag{13}$$

because in this case

$$\Lambda^*_{L,R} = \Xi \Lambda_{L,R} \Xi^{-1}. \tag{14}$$

We finally arrive at

$$\left[\hat{p}^2 - m^2\right] \mathbf{1}_{4\times 4}\, \lambda^{S,A}(p^\mu) = 0, \tag{15}$$

i.e., at the Klein-Gordon equation for each component of $\lambda^{S,A}(p^\mu)$. Why did the acausal solutions disappear? It turns outs that bispinors $\lambda^A(p^\mu) \equiv -\gamma^5 \lambda^S(p^\mu)$ can satisfy the positive-energy equation ($\zeta_\lambda = i$), and bispinors $\lambda^S \equiv -\gamma^5 \lambda^A(p^\mu)$, the negative-energy one ($\zeta_\lambda = -i$), but the dispersion relations will be acausal, Eq. (4), in this unusual case.[3] So, assuming that in the equations (10) one should take $\zeta_\lambda = i$ for describing λ^S and $\zeta_\lambda = -i$, for λ^A we, in fact, implicitly impose mass-shell constraints. The same situation is for the equations (6,7), $u(p^\mu)$ and $v(p^\mu) \equiv \gamma^5 u(p^\mu)$, which can satisfy both the positive- and the negative-energy equations, but the dispersion relations could be unusual.

From a mathematical viewpoint the origin of the appearance of these solutions seems to be related to the properties of the Lorentz transformation operators with respect to the herimitian conjugation operation (see Ref. 14, page 404, for discussion). One should further note that the problem of acausal solutions has interrelations with a mathematical possible situation when operators of the continuous Lorentz transformations are combined with other transformations of the Poincaré group to give $\Lambda_R = -\Lambda_L^{-1}$. Thus, the question whether these solutions would have some physical significance should be solved on the basis of a rigorous analysis of the general structure of the Poincaré transformation group and of the experimental situation in neutrino physics.

Finally, let us mention that another second-order equation in the $(1/2,0) \oplus (0,1/2)$ representation space has been investigated in Ref. 15 and relations with the problem of the lepton mass spectrum have been revealed (see also Refs. 7 and 16).

[3] In the process of the proof one should take into account commutation relations (12) and hence that $\mathcal{S}^+ \gamma^5 \lambda^S(p^\mu) = \lambda^A(p^\mu)$ and $\mathcal{S}^- \gamma^5 \lambda^A(p^\mu) = \lambda^S(p^\mu)$.

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DODATNA RJEŠENJA JEDNADŽBI U $(1/2,0)\oplus(0,1/2)$ REPREZENTACIJSKOM PROSTORU

Analiziraju se disperzijske relacije jednažbi koje je nedavno predložio Ahluwalia za opis neutrina. Jednadžbe za spinove II tipa izvode se na osnovi Wignerovih pravila za lijeve i desne 2-spinove i Ryder-Burgardove relacije. Pokazuje se da jednadžbe sadrže akauzalna rješenja koja su slična onima za jednadžbu sličnu Diracovoj ali drugog reda. Ona se dobiva na sličan način ako se u računu ne primjene ograničenja.