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GENERALIZED JORDAN-WIGNER TRANSFORMATION AND NUMBER OPERATORS

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We define a class of deformed multimode oscillator algebras which possess number operators and can be mapped to multimode Bose algebra by generalized Jordan-Wigner transformation. We construct and discuss the states in the Fock space and the corresponding number operators.

1. Introduction

Recently, much interest has been devoted to the study of quantum groups [1] and to generalizations (deformations) of oscillator algebras [2-6]. Any type of single-mode deformed oscillator can be related (mapped) to a single-mode Bose oscillator [7]. Deformations of multimode oscillators have also been studied [8]. For Pusz-Woronowitz oscillators, covariant under the $SU_q(n)(SU_q(n|m))$ quantum algebra (superalgebra) [2,8] and anyonic-type algebras [9], there exist mappings to multimode Bose algebra. However, deformed generalized quon algebras are known [10] for which such mappings do not exist. All these algebras are associative and the R-matrix approach to them has also been pursued [11]. There exist multimode associative deformed-oscillator algebras that cannot be represented by the R-matrix approach, but can be naturally mapped to the corresponding Bose algebra. We mention that multimode deformed-oscillator algebras are important for possible physical applications, e.g. in q-deformed field theory [9] and generalized statistics [13].

Our aim is to define and analyze a general class of deformed multimode oscillator-algebras which possess well-defined number operators for each type of oscillator and can be mapped to a multimode Bose algebra. These mappings can be viewed as generalized Jordan-Wigner transformations. We discuss the corresponding algebras. For special values of parameters, one recovers all known such algebras [2,3,5,6,8-10]. Finally, we construct and discuss the states in the Fock space and the corresponding number operators for deformed oscillators.

2. *Mappings of the Bose algebra and their Fock-space representation*

Let us define the annihilation and creation operators b_i, b_i^+, $(i \in S)$ satisfying the Bose algebra

$$\begin{aligned} [b_i, b_j^+] &= \delta_{ij}, \quad \forall i, j \in S \\ [b_i, b_j] &= 0 \end{aligned} \tag{1}$$

where $S = 1, 2, ...n$ or S is a set of sites on a lattice. The number operators N_i satisfy

$$\begin{aligned} [N_i, b_j] &= -b_i \delta_{ij}, \quad \forall i, j \in S \\ [N_i, b_j^+] &= b_i^+ \delta_{ij} \\ N_i &= b_i^+ b_i, \quad \forall i, j \in S. \end{aligned} \tag{2}$$

Now we define generalized Jordan-Wigner transformations of the above Bose algebra, Eqs. (1) and (2), as

$$a_i = b_i e^{\sum_j c_{ij} N_j} \sqrt{\frac{\varphi_i(N_i)}{N_i}} \tag{3}$$

where c_{ij} are complex numbers and $\varphi_i(N_i)$ are arbitrary (complex) functions with the properties $\varphi_i(0) = 0$, $\lim_{\epsilon \to 0} \frac{\varphi_i(\epsilon)}{\epsilon} = 1$, $|\varphi_i(1)| = 1, \forall i \in S$. We further assume that $|\varphi_i(N)|$ are bijective, monotonically increasing functions or that $\varphi_i(N) = \frac{1-(-)^N}{2}$, implying $a_i^2 = 0$. The mappings (3) generalize the mappings considered in Refs. 8 and 11. It is important to note that the number operators are preserved, i.e.

$$N_i^{(a)} = N_i^{(b)} \equiv N_i, \quad \forall i \in S. \tag{4}$$

Then it is easy to find the corresponding deformed-oscillator algebra:

$$\begin{aligned} a_i a_j &= e^{c_{ji}-c_{ij}} a_j a_i \quad i{\neq}j \\ a_i a_j^+ &= e^{c_{ij}+c_{ji}^*} a_j^+ a_i, \quad i{\neq}j \\ a_i a_i^+ &= |\varphi_i(N_i+1)| e^{\sum_j (c_{ij}+c_{ij}^*)N_j} e^{(c_{ii}+c_{ii}^*)} \\ a_i^+ a_i &= |\varphi_i(N_i)| e^{\sum_j (c_{ij}+c_{ij}^*)N_j}. \end{aligned} \tag{5}$$

Note that $a_i^2{\neq}0$, unless $b_i\varphi_i(N_i)b_i\varphi_i(N_i) = 0$. For example, if $\varphi_i(N_i) = \frac{1-(-)^{N_i}}{2}$, then $a_i^2 = 0$, implying the hard-core condition for the i^{th} oscillator. Generally, there are other mappings of Bose algebra (Eqs. (1) and (2)), but, in general, they do not have the number operators $N_i^{(a)}$, and Eq. (4) does not hold for mappings other than those in Eq. (3).

We point out that the complete deformed-oscillator algebra is associative owing to the mapping of Bose algebra. The Fock space for the deformed-oscillator algebra is spanned by powers of the creation operators a_i^+, $i \in S$, acting on the vacuum $|0>^{(a)} = |0>^{(b)} \equiv |0>$. The states in the Fock space are specified by the eigenvalues of the number operators N_i, namely $|n_1, n_2, ...n_i.... >^{(a)} = |n_1, n_2, ...n_i.... >^{(b)}$. (If there exists a number $n_i^{(0)} \in \mathbf{N}$, such that $\varphi_i(n_i^{(0)}) = 0$, then $N_i = 0, 1, ...(n_i^{(0)} - 1)$.) States with unit norm are

$$|n_1.....n_n> = \frac{(a_1^+)^{n_1}.......(a_n^+)^{n_n}}{\sqrt{[\tilde{\varphi}_1(n_1)]!.....[\tilde{\varphi}_n(n_n)]!}} e^{-\frac{1}{2}\sum_j \theta_{ji}(c_{ij}+c_{ij}^*)n_i n_j} |0, 0, ...0>$$

$$\begin{aligned} [\tilde{\varphi}(n)]! &= \tilde{\varphi}(n)\tilde{\varphi}(n-1)......\tilde{\varphi}(1) \\ \tilde{\varphi}_i(n_i) &= |\varphi_i(n_i)| e^{(c_{ii}+c_{ii}^*)n_i}, \end{aligned} \tag{6}$$

where θ_{ij} is the step function. (For anyons in $(2 + 1)$-dimension space [9], θ is the angle function.)

Furthermore, the matrix elements of the operators a_i, a_i^+, $i \in S$, are

$$\begin{aligned} &< ....(n_i-1)....|a_i|....n_i... > = < ...n_i....|a_i^+|....(n_i-1).... >^* \\ &= \sqrt{\varphi_i(n_i)} e^{\frac{1}{2}\sum_j (c_{ij}+c_{ij}^*)n_j} \prod_{j\neq i} \delta_{n_j, n_j'}. \end{aligned} \tag{7}$$

We also find for any $k = 0, 1, 2, ...$ that

$$(a_j^+)^k (a_j)^k = \frac{[\tilde{\varphi}_j(N_j)]!}{[\tilde{\varphi}_j(N_j - k)]!} e^{k \sum_{l \neq j} (c_{jl} + c_{jl}^*) N_l}$$

$$(a_j)^k (a_j^+)^k = \frac{[\tilde{\varphi}_j(N_j + k)]!}{[\tilde{\varphi}_j(N_j)]!} e^{k \sum_{l \neq j} (c_{jl} + c_{jl}^*) N_l}. \tag{8}$$

The norms of arbitrary linear combinations of the states in Eq. (6) in the Fock space, corresponding to deformed-oscillator algebra, are positive by definite owing to the mapping of Bose algebra (Eqs. (1) and (2)). Namely, $|n_1, n_2, ...n_i.... >^{(a)} = = |n_1, n_2, ...n_i.... >^{(b)} \equiv |n_1, n_2, ...n_i.... >$.

This class of deformed multimode oscillator algebras comprises multimode Biedenharn-Macfarlaine [3], Aric-Coon [6], two-(p, q) parameter [12], Fermi, generalized Green's [13,14], as well as anyonic [9] and Pusz-Woronowicz (with or without the hard-core condition) oscillators covariant under the $SU_q(n)$ $(SU_q(n|m))$ algebra (superalgebra) [8].

Non-isomorphic (non-equivalent) algebras are classified by different matrix elements given by (7), i.e. with the functions $g_i(n_1, n_2, ...n_n) = |\varphi_i(n_i)| e^{\sum_j (c_{ij} + c_{ij}^*) n_j}$. It is important to mention that there are mappings of Bose algebra which do not preserve the relation $N_i^{(a)} = N_i^{(b)}$, given by (4). Moreover, there are mappings of Bose algebra for which the number operators $N_i^{(a)}$ do not even exist. Such an example is the exchange algebra presented in Ref. 15.

3. Number operators

Finally, we construct and discuss the number operators $N_i^{(a)}$. Let $\varphi_i(n_i)$ be bijective mappings. Then, using Eq. (3), we have

$$b_i = a_i e^{-\sum_j c_{ij} N_j} \sqrt{\frac{N_i}{\varphi_i(n_i)}}, \quad \forall i \in S$$

$$N_i = b_i^+ b_i. \tag{9}$$

The spectra of $N_i^{(a)}$ and $N_i^{(b)}$ coincide. In this case,

$$a_i^+ a_i = g_i(N_1, N_2, ...N_i...) = |\tilde{\varphi}_i(N_i)| e^{\sum_{j \neq i} (c_{ij} + c_{ij}^*) N_j}. \tag{10}$$

Let us denote $a_i^+ a_i = x_i$, and remark that x_i commute with any N_j and among themselves. Then, the N_i operators can be written as

$$N_i = N_i(x_1, ...x_n) = \sum_{k=0}^{\infty} \sum_{(i_1...i_k)} \frac{1}{k!} (\frac{\partial^k N_i}{\partial x_{i_1} ..\partial x_{i_k}})_{x=0} (x_{i_1} x_{i_2} .....x_{i_k})$$

$$N_i(0,0,..0) = 0, \quad \forall i \in S. \tag{11}$$

The coefficients in the Taylor expansion can be obtained from (10)

$$g_i(N_1, N_2, ...N_i...) = x_i, \quad \forall i \in S \tag{12}$$

namely, from

$$(\frac{\partial^k g_i(N_1, N_2, ...N_i...)}{\partial x_{j_1} ..\partial x_{j_k}})_{x=0} = \delta_{k1} \delta_{ji}. \tag{13}$$

These equations give a set of recurrence relations for the coefficients in the Taylor expansion of N_i as a function of the variables x_j, $j \in S$. For example,

$$c_0^{(i)} \equiv (N_i)_{x=0} = 0$$

$$c_j^{(i)} \equiv (\frac{\partial N_i}{\partial x_j})_{x=0} = \frac{1}{(\frac{\partial g_i}{\partial N_i})_0} \delta_{ij} \tag{14}$$

$$c_{jk}^{(i)} \equiv \frac{1}{2} (\frac{\partial^2 N_i}{\partial x_j \partial x_k})_{x=0} = -\frac{1}{2} (\frac{\partial^2 g_i}{\partial N_j \partial N_k})_0 c_i^{(i)} c_j^{(j)} c_k^{(k)}.$$

Specially,

$$c_j^{(i)} = \frac{1}{(\frac{d\varphi_i}{dN_i})_0} \delta_{ij}$$

$$c_{ij}^{(i)} = -\frac{1}{2} (c_{ij} + c_{ji}^*) c_i^{(i)} c_j^{(j)} \tag{15}$$

$$c_{jk}^{(i)} = 0, \quad j \neq i, \quad k \neq i$$

where $g_i(N_1, N_2, ...N_i...)$ is given by Eq. (10).

We also mention the following result. If there exists a continuous mapping defined by (3) from Bose oscillators to deformed oscillators, then the number operators exist and can be written in the form

$$N_i = x_i[c_i^{(i)} + \sum_j c_j^{(i)} x_j + ...] \tag{16}$$

where $x_j = a_j^+ a_j$.

For example, let us consider the n-mode Pusz-Woronowicz oscillator algebra [2] (of Bose type, covariant under the $SU_q(n)$ quantum algebra), $q \in \mathbf{R}$:

$$\begin{gathered} a_i a_j = q^{sgn(j-i)} a_j a_i \\ a_i a_j^+ = q a_j^+ a_i, \quad i \neq j \\ a_i a_i^+ = 1 + (q^2-1) \textstyle\sum_j \theta_{ij} a_j^+ a_j + q^2 a_i^+ a_i. \end{gathered} \tag{17}$$

There exist mappings to the Bose algebra (1) and (2) with $c_{ij} = \theta_{ij} \ln q$, $\varphi(N) = \frac{q^{2N}-1}{q^2-1}$ (see Table 1). Hence, we can calculate the coefficients in the expansion (16). Using Eq. (15), we obtain

$$c_j^{(i)} = \frac{q^2-1}{\ln q^2} \delta_{ij}$$

$$c_{ij}^{(i)} = -\frac{1}{2} \frac{(q^2-1)^2}{\ln q^2} (\delta_{ij} + \theta_{ij}). \tag{18}$$

When $q^2 \to 1$, then $c_i^{(i)} \to 1$, whereas $c_{ij}^{(i)} \to 0$, etc., and $N_i \to a_i^+ a_i$. However, when $q^2 \to 0$, all coefficients diverge and the expansion (16) is not valid. Note that when $q = 0$, the mapping (9) becomes singular, but one can still define the corresponding number operators. For $q = 0$, the Pusz-Woronowicz algebra (17) reduces to

$$\begin{gathered} a_i a_j = 0, \quad i < j \\ a_i a_j^+ = 0, \quad i \neq j \\ a_i a_i^+ = 1 - \textstyle\sum_j \theta_{ij} a_j^+ a_j. \end{gathered} \tag{19}$$

Therefore, we also present the number operators in another form [13] that holds for an arbitrary algebra having number operators, and holds even for $q = 0$:

$$N_i = a_i^+ a_i + \sum_{k=1}^{\infty} \sum_{\pi \in S_k} \sum_{j_1 \ldots j_k} d_{\pi(j_1 \ldots j_k), j_1 \ldots j_k} a_{\pi(j_k)}^+ \ldots\ldots. a_{\pi(j_1)}^+ a_i^+ a_i a_{j_1} \ldots. a_{j_k} \tag{20}$$

where S_k denotes the permutation group and $d_{\pi(j_1\ldots j_k),j_1\ldots j_k}$ are the coefficients of the expansion.

For the Pusz-Woronowicz algebra, the number operators in the above form (see also Ref. 8) are given by

$$N_i = \sum_{k=0}^{\infty}\sum_{j=0}^{k} c_{k,j} \sum_{p_1\ldots.p_j} \theta_{ip_1}.....\theta_{ip_j} a_{p_j}^{+}.....a_{p_1}^{+}(a_i^{+})^{k-j+1}(a_i)^{k-j+1}a_{p_1}......a_{p_j} \tag{21}$$

with the conditions $c_{0,0} = 1$, $c_{0,-j} = 0$, $j > 0$, and with the recurrence relations

$$c_{k+1,k+1-j} = \frac{(1-q^2)(1-q^{2j})}{1-q^{2(j+1)}} c_{k,k+1-j} + q^{2j}(1-q^2)c_{k,k-j}. \tag{22}$$

Starting with $c_{0,0} = 1$, we find that

$$c_{k,1} = c_{k,k} = (1-q^2)^k$$

$$c_{k,0} = \frac{(1-q^2)^{k+1}}{1-q^{2(k+1)}}. \tag{23}$$

In the limit $q^2 \to 1$, the number operators are $N_i = a_i^{+}a_i$. In the limit $q = 0$, all coefficients are $c_{k,j} = 1$, $\forall k, j$. This result is similar to that found by Greenberg in Ref.10 for quons with $q = 0$. Finally, for $\varphi_i(N_i) = \frac{1-(-)^{N_i}}{2}$, the number operators, $N_i^{(a)} \neq N_i^{(b)}$, are simply

$$N_i^{(a)} = a_i^{+}a_i = b_i^{+}b_i[1-\theta(n_i-1)]. \tag{24}$$

We conclude with a few remarks. Using the general Jordan-Wigner transformation, Eq. (3), we unify and generalize the results for the states in the Fock space for the mappings of multimode Bose algebra presented in Table 1. We point out that this transformation represents the most general class of multimode deformed oscillator algebras preserving number operators. We show that for $q \neq 0$ the number operators can be expanded in Taylor series in commuting operators $x_j = a_j^{\dagger}a_j$, Eq. (11), which is complementary to the results of Jagannathan et al. in Ref. 8. However, this expansion (as well as the expansion used in Ref. 8) diverges for $q = 0$. Hence, we give another form of the number operators, Eq. (20), which is valid for arbitrary q and particularly useful for $q = 0$. This expression, Eq. (20), holds for an arbitrary algebra having number operators.

TABLE 1.
Parameters c_{ij} and $\varphi_i(N_i)$ for the various deformed-oscillator algebras.We use definition $[x]_p = \frac{p^x-1}{p-1}$.

type of oscillators	c_{ij}	$\varphi_i(N_i)$
Bose	$i\pi\lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in 2\mathbf{Z}$, $\forall i, j$	N_i
Fermi	$i\pi\lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in 2\mathbf{Z} + 1$, $\forall i, j$	$[N_i]_{-1}$
Green's oscillators [13,14]	$i\pi\lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in \mathbf{Z}$, $\forall i, j$	$[N_i]_{\pm 1}$
Anyonic-type [11]	$i\pi\lambda_{ij}$, $\lambda_{ij} - \lambda_{ji} \in \mathbf{R}$, $\forall i, j$	$[N_i]_{\pm 1}$
Anyons [9]	$i\lambda\theta_{ij}$, $\lambda \in \mathbf{R}$, θ_{ij} is angle	$[N_i]_{\pm 1}$
Pusz-Woronowicz (Bose) [2]	$\theta_{ij} \ln q$, $q \in \mathbf{R}$	$[N_i]_{q^2}$
Pusz-Woronowicz (Fermi) [2]	$\theta_{ij} \ln(-q)$, $q \in \mathbf{R}$	$[N_i]_{-1}$
$SU_q(n\|m)$-covariant (Bose) oscillators [8]	$\theta_{ij} \ln q$, $q \in \mathbf{R}$, $i \le n$	$[N_i]_{q^2}$
$SU_q(n\|m)$-covariant (Fermi) oscillators [8]	$\theta_{ij} \ln(-q)$, $q \in \mathbf{R}$, $n + 1 \le i$	$[N_i]_{-1}$
Biedenharn-Macfarlane [3]	$-\frac{1}{2}\delta_{ij} \ln q_i$, $q_i \in \mathbf{C}$	$[N_i]_{q_i^2}$
Arik-Coon [6]	0, $q_i \in \mathbf{C}$	$[N_i]_{q_i}$

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GENERALIZIRANA JORDAN-WIGNEROVA TRANSFORMACIJA I OPERATORI BROJA ČESTICA

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Definirana je klasa deformiranih oscilatornih algebri sa operatorima broja čestica i poopćenim Jordan-Wignerovim preslikavanjem na Bose algebre. Konstruirana su i diskutirana stanja u Fockovom prostoru te pripadajući operatori broja čestica.