ISSN 1330—0016
CODEN FIZBE7

LETTER TO THE EDITOR

ON BIANCHI-I COSMIC STRINGS

SUDIPTO BHATTACHARYA and TRYAMBAK M. KARADE

Department of Mathematics, Nagpur University Campus, Nagpur 440 010, India

Received 13 November 1991

UDC 530.12

Original scientific paper

Einstein field equations with the cosmological term are solved for an axially symmetric Bianchi-I Letelier string coupled with a magnetic field. It is shown that in the presence of Λ, the initial singularity occurs faster. A formula to obtain numerical value of $\varphi^2 = \frac{A^2}{B^2}$ is given.

The gravitational effects of the gauge cosmic strings have been extensively studied[1-5]. A model of a cloud formed by massive strings was used as a source by Letelier[6] for Bianchi-I and Kantowski-Sachs space-times. The strings forming the cloud were the generalization of the relativistic string model of Takabayashi[7] — the p-strings. The total energy-momentum tensor for a cloud of massive strings is

$$T_\mu^\nu = \varrho V_\mu V^\nu - \lambda x_\mu x^\nu \tag{1}$$

where ϱ is the rest energy density for a cloud of strings with particles attached along the extension. Thus,

$$\varrho = \varrho_p + \lambda \tag{2}$$

where ϱ_p is the particle energy density, λ the tension density of the string, V^μ the 4-vector representing the velocity of the cloud of particles and x^μ, the 4-vector, representing the direction of anisotropy. In this work the Einstein field equations with the cosmological term are solved for an axially symmetric Bianchi-I Letelier string coupled with a magnetic field*, and thereby achieved the generalization of the work of Banerjee et al.[8] (cited henceforth as Ref. 8).

---

*We adopt the signature $(+, -, -, -)$ of the space-time and use units so that $c = \hbar = 1$. We have $V_\mu V^\mu = 1$, $x^\mu x_\mu = -1$, $V_\mu x^\mu = 0$.

An axially symmetric Bianchi-I metric is

$$ds^2 = dt^2 - e^{2\alpha}dx^2 - e^{2\beta}(dy^2 + dz^2) \tag{3}$$

where $\alpha = \alpha(t)$, $\beta = \beta(t)$.

Consider a system of string-dust with a magnetic field along the x-direction as the source for the metric. The energy-momentum tensor for such a system would be given by

$$T^\nu_\mu = T^\nu_{\mu\,string} + E^\nu_{\mu\,mag},$$

where $T^\nu_{\mu\,string}$ is given by (1) and

$$E^\nu_{\mu\,mag} = \frac{1}{4\pi}[-(F^\alpha_\mu F^\nu_\mu) + \frac{1}{4}(F_{\alpha\beta}F^{\alpha\beta})\,\delta^\nu_\mu]. \tag{4}$$

In the co-moving coordinate system, we have

$$T^0_{0\,string} = \varrho, \qquad T^1_{1\,string} = \lambda, \qquad T^\nu_{\mu\,string} = 0 \tag{5}$$

for $\mu, \nu = 2, 3$ and also for $\mu \neq \nu$.

As the magnetic field is along the x-direction alone, F_{23} is the only non-zero component of the Maxwell tensor $F_{\mu\nu}$. From Maxwell equations $F_{[\mu\nu,\sigma]} = 0$ and $[F^{\mu\nu}(\sqrt{-g})]_{,\mu} = 0$, we find F_{23} = constant A. Then

$$E^0_0 = E^1_1 = -E^2_2 = -E^3_3 = -\frac{A^2}{8\pi}e^{-4\beta} \tag{6}$$

(For details, one may refer to Ref. 8).

Now, the Einstein field equations (with the cosmical constant Λ)

$$R^\nu_\mu - \frac{1}{2}Rg^\nu_\mu + \Lambda g^\nu_\mu = -T^\nu_\mu \tag{7}$$

in the conventional notations, assume the following form

$$G^0_0 = 2\dot{\alpha}\dot{\beta} + \dot{\beta}^2 = \varrho + \frac{A^2}{8\pi}e^{-4\beta} + \Lambda \tag{8}$$

$$G^1_1 = 2\ddot{\beta} + 3\dot{\beta}^2 = \lambda + \frac{A^2}{8\pi}e^{-4\beta} + \Lambda \tag{9}$$

$$G^2_2 = G^3_3 + \ddot{\alpha} + \dot{\alpha}^2 + \ddot{\beta} + \dot{\beta}^2 + \dot{\alpha}\dot{\beta} = -\frac{A^2}{8\pi}e^{-4\beta} + \Lambda. \tag{10}$$

Since the number of unknown parameters α, β, ϱ, λ exceeds the number of equations, we close the system by assuming a relation between the metric coefficients given by

$$\alpha = a\beta \tag{11}$$

as has been done in Ref. 8, where a is a constant.

Then (10) reduces to

$$(a+1)\,\ddot{\beta} + (a^2 + a + 1)\,\dot{\beta}^2 = -\frac{A^2}{8\pi}\,\mathrm{e}^{-4\beta} + \Lambda. \tag{12}$$

For $a \neq -1$, Eq. (12) can be written as an integral equation

$$\int \mathrm{d}\left[\dot{\beta}^2\,\mathrm{e}^{2\beta\left(\frac{a^2+a+1}{a+1}\right)}\right] = -\frac{A^2}{4\pi\,(a+1)} \int \left[\mathrm{e}^{2\beta\left(\frac{a^2-a-1}{a+1}\right)} + \right.$$

$$\left. + \frac{2\Lambda}{(a+1)}\,\mathrm{e}^{2\,\left(\frac{a^2+a+1}{a+1}\right)}\right] \mathrm{d}\beta + B,$$

where B is a constant of integration.

Hence,

$$\int \frac{\mathrm{e}^{2\beta}\mathrm{d}\beta}{\left[B\mathrm{e}^{-2\beta\left(\frac{a^2-a-1}{a+1}\right)} - \frac{A^2}{8\pi\,(a^2-a-1)} - \frac{A^2\Lambda}{4\pi\,(a^2+a+1)\,(a+1)}\,\mathrm{e}^{4\beta}\right]^{1/2}} =$$

$$= \pm\,(t - t_0) \tag{13}$$

where t_0 is another constant of integration. In Ref. 8, (13) has been solved for two different cases: $a^2 - a - 1/(a+1) = -1$ and 2. In our case the first condition gives $a = 0$ i. e., $\alpha = 0$ and reduces (12) to

$$\ddot{\beta} + \dot{\beta}^2 = -\frac{A^2}{8\pi}\,\mathrm{e}^{-4\beta} + \Lambda \tag{14}$$

For $y = \dot{\beta}^2$, (14) admits the solution

$$y = B\mathrm{e}^{-2\beta} + \left[\frac{A^2}{8\pi}\,\mathrm{e}^{-4\beta} + \Lambda\right]. \tag{15}$$

Now (13) becomes

$$\int \frac{\mathrm{e}^{2\beta}\mathrm{d}\beta}{\left[B\mathrm{e}^{2\beta} + \frac{A^2}{8\pi} + \Lambda\mathrm{e}^{4\beta}\right]^{1/2}} = \pm\,(t - t_0)$$

which on integration yields

$$e^{2\beta} = \frac{B}{2\Lambda}(\eta \cosh \tau - 1) \tag{16}$$

where

$$\eta = \left(1 - \frac{A^2}{B^2}\frac{\Lambda}{2\pi}\right)^{1/2}, \quad \tau = 2\sqrt{\Lambda}\,(t - t_0).$$

Approximating (16), we obtain

$$e^{2\beta} = p + \Lambda q \tag{17}$$

to first order in Λ, where

$$p = B(t - t_0)^2 - \frac{A^2}{B}\frac{1}{8\pi}$$

$$q = \frac{B}{3}(t - t_0)^4 - \frac{A^2}{B}\frac{1}{4\pi}(t - t_0)^2 - \frac{A^4}{B^3}\frac{1}{64\pi^2}.$$

Eq. (17) shows

$$e^{2\beta}_{ours} = e^{2\beta}_{B} + \Lambda q,$$

where the suffix 'ours' indicates our result and the suffix 'B' indicates that obtained in Ref. 8. In the limit $\Lambda \to 0$, we get the result of Ref. 8.

The proper volume R^3, expansion scalar Θ and shear scalar σ^2 for the metric (4) are, respectively

$$R^3_{ours} = e^{\alpha+2\beta} = e^{2\beta} = \frac{B}{2\Lambda}(\eta \cosh \tau - 1) = R^3_B + \Lambda q \tag{18}$$

$$\Theta_{ours} = V^a{}_{;a} = \dot{\alpha} + 2\dot{\beta} = \frac{1}{R^3_{ours}}\left[\frac{B}{\sqrt{\Lambda}}\,\eta \sin h\tau\right] =$$

$$= \frac{2B}{p}(t - t_0) + \Lambda\left\{\frac{4}{3}\frac{B}{p}(t - t_0)^3 - 2B(t - t_0)\frac{q}{p^2} - \frac{A^2}{B}(t - t_0)\frac{1}{2\pi p}\right\},$$

hence

$$\Theta_{ours} = \Theta_B + \Lambda X \tag{19}$$

where X stands for the quantity in the curly bracket above, and

$$\sigma^2_{ours} = \sigma_{\mu\nu}\sigma^{\mu\nu} = \dot{\alpha}^2 + 2\dot{\beta}^2 - \frac{1}{3}\Theta^2 = \frac{1}{6}\left[\frac{1}{R^3_{ours}}\left(\frac{B}{2\Lambda}\,\eta \sinh \tau\, 2\sqrt{\Lambda}\right)\right]^2,$$

hence

$$\sigma^2_{ours} = \sigma^2_B + \frac{2}{6}\sigma_B \Lambda \left\{\frac{4}{3}\frac{B}{P}(t-t_0)^3 - \frac{A^2}{B}\frac{1}{2\pi p}(t-t_0) - \frac{2B}{p^2}(t-t_0)\,q\right\} \quad (20)$$

to first order in Λ.

Similarly straightforward calculations to first order in Λ yield

$$\varrho_{ours} = \varrho_B + \Lambda\,(Y) \quad (21)$$

and

$$\lambda_{ours} = \lambda_B + \Lambda\,(Z) \quad (22)$$

where ϱ_B and λ_B are given in Ref. 8 and

$$Y = \frac{4}{3}B^2(t-t_0)^4\frac{1}{p^2} - B^2\frac{2q}{p^3}(t-t_0)^3 - \frac{A^2}{8\pi}\frac{1}{p^2}(t-t_0)^2 - 1$$

$$Z = \frac{2Bq}{p^2} + \frac{4B}{p}(t-t_0)^2 - \frac{A^2}{B}\frac{1}{2\pi p} - \frac{A^2}{2\pi p^2}(t-t_0)^2 - \frac{2B}{p^3}(t-t_0)^2 -$$

$$-\frac{3}{4}\frac{A^2}{\pi}\frac{q}{p^3} - 1.$$

In our case, the condition for singularity in the initial epoch i. e. $R^3 \to 0$ gives

$$T^2_{ours} = (t-t_0)^2 = \frac{A^2}{8\pi B^2} - \frac{\Lambda}{24\pi^2}\frac{A^4}{B^4} \quad (23)$$

to first order in Λ. Here we find

$$T_{ours} - T_B = -ve.$$

Hence the introduction of Λ in the string-filled universe speeds up the occurrence of singularity.

The tension density λ of the string vanishes at the instant specified by the cubic equation

$$\left(\frac{2}{3}\Lambda B^3 + 4B^3\Lambda - \frac{A^2}{2\pi}B^2\Lambda - B^3\Lambda\right)T^3 +$$

$$+\left(B^3 - \frac{2}{3}\Lambda B^2 C - 8B^2 C\Lambda - \frac{A^2}{2\pi}\Lambda B + \frac{A^2}{\pi}BC\Lambda - \frac{A^2}{4\pi}B\Lambda\right)T^2 +$$

$$+\left(4BC^2\Lambda - \frac{3A^2}{8\pi}B - CB^2 - 4B^2C\Lambda - 5C^2B\Lambda + \frac{A^2}{\pi}AC - \right.$$

$$\left. - \frac{A^2}{2\pi}C^2\Lambda - 2B^2\Lambda\right) T + \left(\frac{3A^2C}{8\pi} + \right.$$

$$\left. + 4BC^2\Lambda - \frac{A^2}{B}\frac{\Lambda}{2\pi}C^2 + \frac{3A^2C\Lambda}{2\pi} + C^3\Lambda\right) = 0. \tag{24}$$

One of the roots of (24), in the limit $\Lambda \to 0$, reduces to the condition

$$T^2 = (t - t_0)^2 = \frac{3A^2}{8\pi B^2},$$

which is the corresponding value obtained in Ref. 8.

Another result of significance is regarding η, which we require to be real on physical grounds. Hence,

$$\eta^2 = \frac{B^2}{\Lambda} - \frac{A^2}{2\pi} \geq 0$$

i. e.

$$\Lambda \leq 2\pi\varphi^2, \quad \text{where} \quad \varphi^2 = \frac{A^2}{B^2}.$$

Therefore

$$\varphi \geq \pm \sqrt{\frac{\Lambda}{2\pi}} \tag{25}$$

which gives the numerical estimation of the ratio φ. In absence of the magnetic field, φ becomes physically meaningless.

References

1) A. Vilenkin, Phys. Rev. D **23** (1981) 852; Phys. Rev. Lett. **46** (1981) 1169, 1496 (E); Phys. Lett. **107B** (1981) 47;
2) J. R. Gott, Ap. J. **288** (1985) 422;
3) D. Garfinkle, Phys. Rev. D **32** (1985) 986;
4) W. A. Hiscock, Phys. Rev. D **31** (1985) 3288;
5) T. Vachaspati and A. Vilenkin, Phys. Rev. D **31** (1985) 3052;
6) P. S. Letelier, Phys. Rev. D **28** (1983) 2414;
7) T. Takabayashi, *Quant. Mech*; *Determinism, Causality and Particles,* ed. M. Flato (Holland: Reidel Dorbrecht) p. 179 (1978);
8) A. Banerjee, A. K. Sanyal and S. Chakraborty, Pramana — J. Phys. **34** (1990) 1.

O BIANCHI-I KOZMIČKIM STRUNAMA

SUDIPTO BHATTACHARYA i TRYAMBAK KARADE

Dept. of Mathematics, Nagpur University Campus, Nagpur 440 010, India

UDK 530.12

Originalni znanstveni rad

Riješena je Einsteinova jednadžba polja s kozmološkim članom za aksijalno-simetričnu Bianchi-I Letelierovu strunu vezanu za magnetsko polje. Pokazano je da se u prisustvu Λ početni singularitet brže pojavljuje.