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CLASSICAL AND QUANTUM COSMOLOGY OF THE SÁEZ-BALLESTER THEORY

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We study the generalization of the Sáez-Ballester theory for a flat FRW cosmology. Classical solutions are obtained by using the Hamilton-Jacobi approach. Contrary to claims in the literature, it is shown that the Sáez-Ballester theory cannot provide a realistic solution to the dark matter problem. Furthermore, the quantization procedure of the theory can be simplified by reinterpreting the theory in the Einstein frame, where the scalar field can be considered as part of the matter content of the theory. Finally, exact solutions are found for the Wheeler-DeWitt equation.

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1. *Introduction*

The inclusion of scalar fields to homogeneous cosmologies is a typical practice in different scenarios, such as inflation, dark matter, and dark energy [1]. Since the early seventies, the problem of finding the appropriate sources of matter and its corresponding Lagrangian to describe particular scenarios has been studied [2, 3]. An interesting approach was presented by Sáez and Ballester (SB) [4], a scalar-tensor theory of gravity where the metric is coupled to a dimensionless scalar field.

Despite the dimensionless character of the scalar field, an antigravity regime appears, and has been proposed as a possible way to solve the missing-matter problem in non-flat Friedmann-Robertson-Walker (FRW) cosmologies [5, 6, 7, 8]. Also, Armendariz-Picon et al. called this scenario as *K-essence* [9], which is characterized by a scalar field with a non-canonical kinetic energy.

Usually, K-essence models are restricted to a lagrangian density of the form

$$S = \int \mathrm{d}^4x \sqrt{-g}\, f(\phi)\, (\nabla\phi)^2 \,. \tag{1}$$

One of the motivations to consider this type of lagrangian originates from string theory [10]. For more details about the K-essence applied to dark energy, you can see [1] and references therein. Recently, this formalism was employed in cosmic strings where a general linear equation of state of the cosmic string tension density was used with the proper energy density of the universe, obtaining solutions [11, 12]. On another front, the quantization program of this theory has not been constructed, the main complication can be traced to the lack of an ADM type formalism. We can transform this theory to conventional one where the dimensionless scalar field is obtained from energy-momentum tensor as a exotic matter contribution, and in this sense we can use this formalism for the quantization program, where the ADM formalism is well known [2].

In this paper, we study a generalization of the SB theory and its transformation into a conventional tensor theory, where the dimensionless scalar field is interpreted as exotic matter [2] in the energy-momentum tensor. We obtain the corresponding canonical lagrangian $\mathcal{L}_{\rm can}$ for a FRW metric and calculate the classical hamiltonian $\mathcal{H}$, from which we find the Wheeler-DeWitt (WDW) equation of the corresponding cosmological model. As we shall see, the general behaviour of the scalar field as a function of the scale factor corresponds to stiff matter and not to dust. For this reason, the missing-matter problem is not solved, as proposed in the original approach. Finally, the classical and quantum solutions for a barotropic perfect fluid, using the Hamilton-Jacobi formalism, are found.

The paper is arranged as follows, In Section 2 we write the generalized SB formalism in the usual manner, and we also present solutions to some models. In Section 3, using the transformation and the Hamiltonian constraint $\mathcal{H}$, we find the Wheeler-DeWitt (WDW) equation and its solution of the corresponding cosmological model under study. Section 4 is devoted to conclusions and outlook.

2. *Generalized Sáez-Ballester theory*

The simplest generalization of the Sáez-Ballester theory [4] with a cosmological constant term is

$$\mathcal{L}_{\rm geo} = \left(R - 2\lambda - F(\phi)\phi_{,\gamma}\phi^{,\gamma}\right) , \tag{2}$$

where R is the Ricci scalar, $\phi^{,\gamma} = g^{\gamma\alpha}\phi_{,\alpha}$, and $F(\phi)$ is a dimensionless and arbitrary function of the scalar field. According to common wisdom, the Lagrangian (2) can be interpreted a scalar field theory without scalar potential but with an exotic kinetic term. The complete action is

$$I = \int_\Sigma \sqrt{-g}(\mathcal{L}_{\rm geo} + \mathcal{L}_{\rm mat})\, \mathrm{d}^4x \,, \tag{3}$$

where we have included a matter Lagrangian $\mathcal{L}_{\rm mat}$, and g is the determinant of metric tensor. The field equations derived from the above action are

$$G_{\alpha\beta} + g_{\alpha\beta}\lambda - F(\phi)\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}\right) = 8\pi G T_{\alpha\beta}\,, \tag{4a}$$

$$2F(\phi)\phi^{,\alpha}_{;\alpha} + \frac{\mathrm{d}F}{\mathrm{d}\phi}\phi_{,\gamma}\phi^{,\gamma} = 0\,, \tag{4b}$$

in which G is the gravitational constant, and a semicolon represents a covariant derivative.

The same set of equations (4a) and (4b) is obtained if we consider the scalar field ϕ as part of the matter content, $\mathcal{L}_\phi = F(\phi)g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. In this line of reasoning, action (3) can be rewritten as a geometrical part (Hilbert-Einstein with λ) and matter content (usual matter plus a term that corresponds to the scalar field component of SB theory),

$$I = \int\limits_\Sigma \sqrt{-g}\,(R - 2\lambda + \mathcal{L}_{\rm mat} + \mathcal{L}_\phi)\ \mathrm{d}^4x\ . \tag{5}$$

The two theories are equivalent at the classical level, hence we can propose that this equivalence is also valid at the quantum level as only the hamiltonian constraint is modified [2].

Using this action, we obtain the Hamiltonian for SB. Let us start with the line element for a homogeneous and isotropic FRW universe,

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\mathrm{d}\Omega^2\right], \tag{6}$$

where $a(t)$ is the scale factor, $N(t)$ is the lapse function, and κ is the curvature constant that can take the values 0, 1 and -1, for flat, closed and open universe, respectively. The total Lagrangian density then reads

$$\mathcal{L} = \frac{6\dot{a}^2 a}{N} - 6\kappa N a + \frac{F(\phi)a^3}{N}\dot{\phi}^2 + 16\pi G N a^3\rho - 2Na^3\lambda\,, \tag{7}$$

where ρ is the matter energy density. We will assume that it complies with a barotropic equation of state of the form $p = \gamma\rho$, where γ is a constant. The conjugate momenta are

$$\begin{aligned}\Pi_a &= \frac{\partial\mathcal{L}}{\partial\dot{a}} = \frac{12a\dot{a}}{N}, \qquad \rightarrow \qquad \dot{a} = \frac{N\Pi_a}{12a}\,,\\ \Pi_\phi &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{2Fa^3\dot{\phi}}{N}\,, \qquad \rightarrow \qquad \dot{\phi} = \frac{N\Pi_\phi}{2Fa^3}\,.\end{aligned} \tag{8}$$

From the canonical form of the Lagrangian density (7), and the solution for the barotropic fluid equation of motion, we find the Hamiltonian density for this theory

$$\mathcal{H} = \frac{a^{-3}}{24}\left[a^2\Pi_a^2 + \frac{6}{F(\phi)}\Pi_\phi^2 + 144\kappa a^4 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)}\right], \tag{9}$$

where ρ_γ is an integration constant.

2.1. Classical solutions for flat FRW

Using the transformation $\Pi_q = \mathrm{d}S_q/\mathrm{d}q$, the Einstein-Hamilton-Jacobi (EHJ) associated to Eq. (9) is

$$a^2\left(\frac{\mathrm{d}S_a}{\mathrm{d}a}\right)^2 + \frac{6}{F(\phi)}\left(\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi}\right)^2 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} = 0\,. \tag{10}$$

The EHJ equation can be further separated into the equations

$$\frac{6}{F(\phi)}\left(\frac{\mathrm{d}S_\phi}{\mathrm{d}\phi}\right)^2 = \mu^2\,, \tag{11}$$

$$a^2\left(\frac{\mathrm{d}S_a}{\mathrm{d}a}\right)^2 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} = -\mu^2\,, \tag{12}$$

where μ is a separation constant. With the help of Eqs. (8), we can obtain the solution up to quadratures of Eqs. (11) and (12),

$$\int\sqrt{F(\phi)}\,\mathrm{d}\phi = \frac{\mu}{2\sqrt{6}}\int a^{-3}(\tau)\,\mathrm{d}\tau\,, \tag{13a}$$

$$\Delta\tau = \int\frac{a^2\mathrm{d}a}{\sqrt{\frac{8}{3}\pi G\rho_\gamma a^{3(1-\gamma)} - \frac{\lambda}{3}a^6 - \nu^2}}\,, \tag{13b}$$

with $\nu = \mu/12$. Eq. (13a) readily indicates that

$$F(\phi)\dot{\phi}^2 = 6\nu^2 a^{-6}(\tau)\,. \tag{14}$$

Also, this structure is directly obtained for this model solving Eq. (4b). Moreover, the matter contribution of the SB scalar field to the r.h.s. of the Einstein equations would be

$$\rho_\phi = \frac{1}{2}F(\phi)\dot{\phi}^2 \propto a^{-6}. \tag{15}$$

The contribution of the scalar field is the same as that of stiff matter with a barotropic equation of state $\gamma = 1$. This is an interesting result, since the original SB theory was thought of as a form to solve the missing matter problem now generically called the dark matter problem. To solve the latter, one needs a fluid behaving as dust with $\gamma = 0$. It is surprising that such a general result remained unnoticed until now in the literature about SB.

Also, we have identified the general evolution of the scalar field with that of a stiff fluid what means that Eq. (13b) can be integrated separately without a complete solution for the scalar field. For completeness, we give below a compilation of exact solutions in the case of the original SB theory.

If $F(\phi) = \omega\phi^m$, then we have two cases that correspond to $m = -2$ and $m \neq -2$; the general solution for the scalar field is

$$\phi = \begin{cases} \exp\left[\dfrac{6\nu}{\sqrt{6\omega}} \displaystyle\int a^{-3}(\tau)\mathrm{d}\tau\right] & m = -2, \\ \left[\dfrac{2\nu(m+2)}{\sqrt{6\omega}} \displaystyle\int a^{-3}(\tau)\mathrm{d}\tau\right]^{2/(m+2)} & m \neq -2. \end{cases} \tag{16}$$

These equations do not have a general solution, but we can solve them for the particular values of γ, with which we can solve Eq. (13b) for the scale factor a. The general results are presented as the following two cases. The solutions are for the volume function and field ϕ in the scenarios $\gamma = -1, 0$. The solution for $\gamma = 1$ is similar to $\gamma = -1$, only the constants are re-scaled.

Case I. Inflation, $\gamma = -1$.

$$a^3(\tau) = \frac{1}{2b_{-1}}\left(\mathrm{e}^{3\sqrt{b_{-1}}\,\Delta\tau} + b_{-1}\nu^2\mathrm{e}^{-3\sqrt{b_{-1}}\,\Delta\tau}\right)$$

where

$$b_{-1} = \frac{8}{3}\pi G\rho_{-1} - \frac{\lambda}{3}$$

For $m = -2$, we obtain

$$\phi(\tau) = \exp\left[\frac{4}{\sqrt{6\omega}}\arctan\left(\frac{\exp[3\sqrt{b_{-1}}\Delta\tau]}{\nu\sqrt{b_{-1}}}\right)\right]$$

while for $m \neq -2$, the results is

$$\phi(\tau) = \left[\frac{2(m+2)}{\sqrt{6\omega}}\arctan\left(\frac{\exp[3\sqrt{b_{-1}}\Delta\tau]}{\nu\sqrt{b_{-1}}}\right)\right]^{2/(m+2)}$$

Case II. Dust, $\gamma = 0$.

$$a^3(\tau) = \frac{3}{4\sqrt{3|\lambda|}}\mathrm{e}^{-\sqrt{3|\lambda|}\tau}\left[4\nu^2 + \left(\mathrm{e}^{\sqrt{3|\lambda|}\tau} - \frac{3b_0}{\sqrt{3|\lambda|}}\right)^2\right]$$

where

$$b_0 = \frac{8}{3}\pi G\rho_0,\; |\lambda| > 0$$

For $m = -2$, we obtain

$$\phi(\tau) = \exp\left[\frac{4}{\sqrt{6}\,\omega}\arctan\left(\frac{\sqrt{3|\lambda|}\exp[\sqrt{3|\lambda|}\Delta\tau] - 3b_0}{2\nu\sqrt{3|\lambda|}}\right)\right]$$

while for $m \neq -2$, the results is

$$\phi(\tau) = \left[\frac{4(m+2)}{\sqrt{6\omega}}\arctan\left(\frac{\sqrt{3|\lambda|}\exp[\sqrt{3|\lambda|}\Delta\tau] - 3b_0}{2\nu\sqrt{3|\lambda|}}\right)\right]^{2/(m+2)}$$

In Fig. 1 one can see that the function ϕ asymptotically goes to a constant value. This constant value depends on the value of $4/\sqrt{6\omega}$.

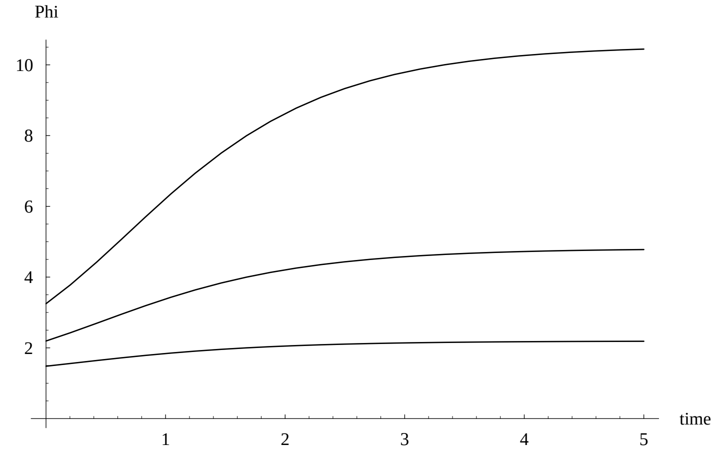

Fig. 1. Plot of the function ϕ for the inflationary case and $m = -2$. The lines correspond to different values of $4/\sqrt{6\omega} = 0.5,\ 1,\ 1.5$.

In Fig. 2 the behaviour of the scalar function ϕ depends strongly to value on the parameter m. For negative values, as in the previous case, ϕ asymptotically approaches a constant value that depends on m.

The classical solutions when $F(\phi) = \omega \mathrm{e}^{m\phi}$ have the following structure

$$\phi(\tau) = \frac{2}{m}\ln\left[\frac{m}{2}\sqrt{\frac{6\nu^2}{\omega}}\int a^{-3}(\tau)\mathrm{d}\tau + \mathrm{e}^{(m/2)\phi_0}\right]. \tag{17}$$

The solutions above were checked to satisfy the Einstein field equations encoded in Eqs. (4b), using the REDUCE 3.8 package.

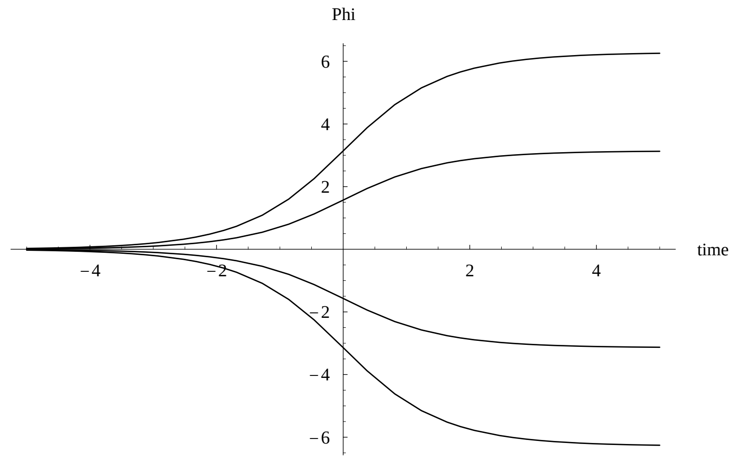

Fig. 2. Plot of the function ϕ, for the inflationary case and $m \neq -2$. The plots correspond to different values of $m = -4, -2, 2, 4$.

3. *Quantum FRW cosmological model*

One of open problems of SB is the lack of a quantization procedure. In this section, we use canonical quantization for the models of the previous section. By the usual representation for the momenta operators $\Pi_q = -\mathrm{i}\partial/\partial q$, ($\hbar = 1$), we obtain the WDW equation

$$\Bigg[-a^2\frac{\partial^2}{\partial a^2} - qa\frac{\partial}{\partial a} - \frac{6}{F(\phi)}\frac{\partial^2}{\partial \phi^2} - \frac{6s}{F(\phi)}\phi^{-1}\frac{\partial}{\partial \phi}$$

$$+144\kappa a^4 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)}\Bigg]\Psi = 0, \tag{18}$$

where q and s are real constants that measure the ambiguity in the factor ordering in the operators Π_a and Π_ϕ, while Ψ is the wave function for this cosmological model. Using $\Psi(a,\phi) = \mathcal{A}(a)\mathcal{B}(\phi)$, Eq. (18) gives the equations

$$-a^2\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}a^2} - qa\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}a} + \left(144\kappa a^4 + 48a^6\lambda - 384\pi G\rho_\gamma a^{3(1-\gamma)} - \mu^2\right)\mathcal{A} = 0, \tag{19}$$

$$\phi\frac{\mathrm{d}^2\mathcal{B}}{\mathrm{d}\phi^2} + s\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}\phi} - \frac{\mu^2}{6}\phi F(\phi)\mathcal{B} = 0. \tag{20}$$

Equation (19) does not have a general solution for κ. We work with $\kappa = 0$ and particular values of the γ parameter.

When $\gamma = -1$, the exact solution is

$$\mathcal{A}(a) = a^{\pm 3\sqrt{\nu^2+(1/9)\mu^2}}\ Z_\nu \left(\frac{\sqrt{b}}{3} a^3 \right), \tag{21}$$

where $\nu = \frac{1}{6}\sqrt{(1-q)^2 - 4\mu^2}$ and $b = 384\pi G\rho_{-1} - 48\lambda$. We can see that when $b > 0$, the generic Bessel function $Z_\nu \to J_\nu$, and when $b < 0$, $Z_\nu \to (K_\nu, I_\nu)$

Another solvable case is $\gamma = 1$, the solution is the same and the changes appear in the constants $\mu^2 \to 384\pi G\rho_1 + \mu^2$ and $b = -48\lambda$.

To solve Eq (20), we apply this approach to SB theory. The case when $m \neq -2$ [13] is written in terms of generic Bessel functions Z_η as

$$B(\phi) = c\phi^{((m+2)/2)\eta}\ Z_\eta \left(\frac{2\sqrt{-\xi}}{m+2} \phi^{(m+2)/2} \right), \tag{22}$$

where c is an integration constant, and $\eta = (1-s)/(m+2)$, $\xi = \mu^2\omega/6$. Also, we can see that the generic Bessel function $Z_\eta \to J_\eta$ when $\omega < 0$, or (K_η, I_η) when $\omega > 0$.

We can build the wave packet

$$\Psi_{\eta\nu} = \int\limits_\eta \int\limits_\nu \mathcal{F}(\eta)\mathcal{G}(\nu)\phi^{((m+2)/2)\eta}\ Z_\eta \left(\frac{2\sqrt{-\xi}}{m+2} \phi^{(m+2)/2} \right) a^{\pm 3\sqrt{\nu^2+\frac{1}{9}\mu^2}}\ Z_\nu \left(\frac{\sqrt{b}}{3} a^3 \right) \mathrm{d}\eta \mathrm{d}\nu \tag{23}$$

For particular values of the constant m, the exact solutions are very simple. For instant when $m = -2$, we have the Euler equation whose solution is

$$\mathcal{B}(\phi) = \phi^{(1-s)/2} \begin{cases} [c_1\phi^\alpha + c_2\phi^{-\alpha}] & (1-s)^2 > 4b \\ [c_1 + c_2 \ln\phi] & (1-s)^2 = 4b \\ [c_1 \sin(\alpha\ln\phi) + c_2\cos(\alpha\ln(\phi))] & (1-s)^2 < 4b \end{cases} \tag{24}$$

with $\alpha = \frac{1}{2}\sqrt{(1-s)^2 - 4b}$ and $b = -\omega\mu^2/6$.

When $m = -6$ and $s = -1$, the transformations $z = \phi^{-2}$ and $B = u/z$, lead to $4\mathrm{d}^2u/\mathrm{d}z^2 - \mu^2(\omega/6)u = 0$, with solutions

$$u(z) = \begin{cases} c_1 \sinh\left(\sqrt{\frac{\mu^2\omega}{24}} z\right) + c_2 \cosh\left(\sqrt{\frac{\mu^2\omega}{24}} z\right) & \omega > 0 \\ c_1 \sin\left(\sqrt{\frac{\mu^2\omega}{24}} z\right) + c_2 \cos\left(\sqrt{\frac{\mu^2\omega}{24}} z\right) & \omega < 0 \end{cases} \tag{25}$$

or, in the original variables

$$\mathcal{B}(\phi) = \phi^2 \begin{cases} c_1 \sinh\left(\sqrt{\frac{\mu^2\omega}{24}}\frac{1}{\phi^2}\right) + c_2 \cosh\left(\sqrt{\frac{\mu^2\omega}{24}}\frac{1}{\phi^2}\right) & \omega > 0 \\ c_1 \sin\left(\sqrt{\frac{\mu^2\omega}{24}}\frac{1}{\phi^2}\right) + c_2 \cos\left(\sqrt{\frac{\mu^2\omega}{24}}\frac{1}{\phi^2}\right) & \omega < 0 \end{cases} \tag{26}$$

4. *Conclusions*

We studied the generalization of the SB theory by including a dimensionless function of the scalar field $F(\phi)$. The classical dynamics of the theory was obtained from the corresponding classical Lagragian and Hamiltonian densities; the solutions are given up to quadratures.

One general result here is that the evolution of the scale factor of the Universe does not depend upon the particular form of the function $F(\phi)$. The contribution of the scalar field in the SB theory is that of perfect fluid with a stiff (barotropic) equation of state. For this reason its contribution to the matter budget of the Universe is at early times.

Another conclusion is that SB, whether in its original form as given in Ref. [4], or in the generalized case, cannot be a solution to the dark matter riddle of cosmology.

In the quantum regime, it was necessary to build an equivalent lagrangian density in order to apply canonical quantization. We checked this approach using the original Sáez-Ballester formalism, obtaining the exact solutions in both regimes, classical and quantum for particular values of the γ parameter. This formalism will be used with anisotropic cosmological models, and will be reported elsewhere.

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KLASIČNA I KVANTNA KOZMOLOGIJA SÁEZ-BALLESTEROVE TEORIJE

Proučavamo poopćenje Sáez-Ballesterove teorije za ravnu FRW kozmologiju. Postigli smo klasična rješenja primjenom Hamilton-Jacobijeve metode. Suprotno tvrdnjama u ranijim radovima, pokazujemo da Sáez-Ballesterova teorija ne daje realno rješenje pitanja tamne tvari. Nadalje, postupak kvantizacije teorije može se pojednostaviti novim pristupom teoriji prema Einsteinu kada se skalarno polje tumači kao dio tvari u svemiru. Konačno, našli smo egzaktna rješenja Wheeler-DeWittove jednadžbe.