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LETTER TO THE EDITOR

FRACTIONAL ACTION-LIKE VARIATIONAL APPROACH, PERTURBED EINSTEIN'S GRAVITY AND NEW COSMOLOGY

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Fractional calculus is a powerful tool for extending concepts of classical physics. The combination of fractional calculus and cosmology is an especially promising strategy to find new insights and answers to actual problems in astronomy and large-scale physics. In this paper, the fractional action-like variational approach, introduced by the author in 2005, is used to investigate some new cosmological features. We discuss in particular a modified cosmological constant from fractional action-like variational approach with perturbed Einstein's gravity, resulting in a dissipative cosmological constant.

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Fractional integrals and derivatives, known as fractional calculus, play an important and leading role in the understanding of complex classical and quantum dynamical systems [1 − 5]. They have been applied successfully in different fields of science. Physicists and mathematicians have begun to explore the realm of applications of fractional calculus with ever new developments rapidly taking place in diverse fields of science. For an excellent list of works and a historical survey in this area, the reader in refereed to the bibliography prepared by Ross and reprinted in the monograph by Oldham and Spanier [1]. The fractional calculus of variations (CoV) and correspondent fractional Euler-Lagrange equations (FELE) is one of the most and significant topics encountered in fractional calculus [6 − 31]. The FELE has been studied in recent years and its applications in treating a variety of problems have been broadly addressed.

In an attempt to model nonconservative and weak dissipative dynamical systems, we proposed in recent work [16, 17] a novel dimensional variational approach known as the fractional action-like variational approach (FALVA) where fractional time integral introduces only one fractional parameter $\alpha > 0$, while in other models an arbitrary number of fractional parameters (orders of derivatives) appear. The novelty in the Euler-Lagrange equation is the presence of fractional generalized external force acting on the system, while in other models an arbitrary number of fractional parameters *(orders of derivatives)* appears. Some important features and implications of FALVA in classical, quantum, geometrical dynamics and cosmology are discussed in details in the series of papers [16 − 38].

In this letter, we discuss another nice application of FALVA in Einstein's general relativity. The main goal of the present work is to explore some new consequences not found within the standard formalism of Einstein's theory of general relativity (ETGR) which may have interesting consequences in high-energy physics.

Within the framework of FALVA, the ETGR is modified as follows: consider in the frame (x, y, z, T), where $T = \tau - t$, the classical Lagrangian of a curve defined in general by $L = (1/2)g_{ij}\dot{x}^i\dot{x}^j$, where $\dot{x} = \mathrm{d}\tau/\mathrm{d}t$, g_{ij} is the metric tensor and m is the particle mass. The fractional action of the theory is defined within the framework of FALVA by [17, 34]

$$S = -\frac{m}{2\Gamma(\alpha)} \int \dot{x}^i \dot{x}^j g_{ij}(x)(t-\tau)^{\alpha-1} \mathrm{d}\tau \,. \tag{1}$$

$\Gamma(\alpha) = \int\limits_0^\infty \exp t^{\alpha-1} \exp(-t)\mathrm{d}t$ is the Euler gamma function, $0 < \alpha \leq 1$, τ is the intrinsic time and t is the observer time, $t \neq \tau$.

The corresponding fractional Euler-Lagrange equation derived from the fractional action (1) is

$$\frac{\mathrm{d}^2 x^l}{\mathrm{d}\tau^2} + \frac{\alpha-1}{\tau-t}\frac{\mathrm{d}x^l}{\mathrm{d}\tau} + \Gamma^l_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = 0 \,, \tag{2}$$

where Γ^l_{ij} are the Christoffel connection coefficients of the metric g_{ij} defined by

$$\Gamma^l_{ij} = \begin{Bmatrix} l \\ ij \end{Bmatrix} = \frac{1}{2} g^{l\sigma}\left[g_{j\sigma,i} + g_{i\sigma,j} + g_{ij,\sigma}\right]. \tag{3}$$

Here $g_{ij,\sigma} = \partial g_{ij}/\partial x^\sigma$ and so on. Equation (2) is the modified fractional geodesic equation. To see the effect of Eq. (2) on ETGR, we assume that the test particle's velocity is roughly zero and that the metric and its derivative are more or less static and that the squares of deviations from the Minkowski metric are insignificant. Applying these simplifying hypotheses to the spatial components of the fractional geodesic equation gives

$$\mathrm{div}\,\vec{\gamma} + \frac{\alpha-1}{\tau-t}\partial_i v^i + R_{00} = 0 \,. \tag{4}$$

$v^i = \mathrm{d}x^i/\mathrm{d}\tau$ is the particle velocity and $\mathrm{div}\vec{\gamma} = \partial_i \gamma^i = -4\pi G\rho$ is the Poisson equation with $\gamma^i = \mathrm{d}v^i/\mathrm{d}\tau$, the particle acceleration, ρ is the mass density in gravitational interaction and G the Newton gravitational constant. R_{00} is the time component of the Ricci tensor defined usually by

$$R_{\mu\nu} = \partial_\rho \begin{Bmatrix} \rho \\ \mu\nu \end{Bmatrix} - \partial_\nu \begin{Bmatrix} \rho \\ \mu\rho \end{Bmatrix} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \begin{Bmatrix} \rho \\ \lambda\rho \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \mu\rho \end{Bmatrix} \begin{Bmatrix} \rho \\ \nu\lambda \end{Bmatrix}, \tag{5}$$

from which we obtain effortlessly

$$R_{00} = \partial_\rho \begin{Bmatrix} \rho \\ 00 \end{Bmatrix} - \partial_0 \begin{Bmatrix} \rho \\ 0\rho \end{Bmatrix} + \begin{Bmatrix} \lambda \\ 00 \end{Bmatrix} \begin{Bmatrix} \rho \\ \lambda\rho \end{Bmatrix} - \begin{Bmatrix} \lambda \\ 0\rho \end{Bmatrix} \begin{Bmatrix} \rho \\ 0\lambda \end{Bmatrix}, \tag{6}$$

Our simplifying assumptions make the squares of $\{\}$ disappear jointly with the time derivatives and hence

$$R_{00} = \partial_i \begin{Bmatrix} i \\ 00 \end{Bmatrix} \tag{7}$$

From Eq. (4) it follows that

$$R_{00} = 4\pi G\rho \left(1 + \frac{1-\alpha}{4\pi G\rho T} \partial_i v^i\right) \equiv 4\pi\rho G_{\mathrm{effective}}\,, \tag{8}$$

where

$$G_{\mathrm{effective}} = G\left(1 + \frac{1-\alpha}{4\pi G\rho T} \partial_i v^i\right) \equiv G + \Delta G_\alpha \tag{9}$$

is the effective gravitational coupling constant and $\Delta G_\alpha = G \dfrac{(1-\alpha)}{4\pi\rho(t-\tau)} \partial_i v^i$. When $\alpha = 1$, $G_{\mathrm{effective}} = G_\infty \equiv G(\tau \to \infty)$ as it is expected. The Einstein's field equations are, therefore, modified and may be written in the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi (G + \Delta G_\alpha) T_{\mu\nu}\,, \tag{10}$$

where $R_{\mu\nu}$ is the Riemann tensor, R the Ricci tensor and $T_{\mu\nu}$ is the stress energy tensor.

We expect now some important consequences in the Friedmann-Roberston-Walker (FRW) cosmology with the presence of the cosmological constant Λ. In the traditional Einstein general relativity, Λ is fixed, and it serves as the source for the metric field. In other words, the input in the Einstein field equation is Λ, the output is de-Sitter expansion, if matter is absent. The Einstein field equation does not allow us to obtain the time dependence of the cosmological constant because of the Bianchi identities $G^{\nu}_{u:\nu} = 0$, $G_{u\nu} \equiv R_{u\nu} - (1/2) g_{u\nu} R$ and covariant conservation law of matter $T^{\nu\mathrm{matter}}_{u:\nu} = 0$, both leading to $\partial_u \Lambda = 0$. But they allow us to obtain the value of the cosmological constant in different static universes, such as the Einstein closed universe, where the cosmological constant is obtained as a function of the curvature and matter density.

To describe the evolution of the cosmological constant in a flat and four-dimensional FRW spacetime manifold described by the Friedmann-Robertson-Walker (FRW) metric

$$\mathrm{d}s_4^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad (i,j=1,2,3)\,, \tag{11}$$

where $a(t)$ is the scale factor, one needs to introduce the relaxation term or a dissipative one. For this we use the Raychaudhuri's expansion scalar factor $\partial_i v_i \equiv 3\dot{\theta} = 3\dot{a}/a$, $\dot{a} = \mathrm{d}a/\mathrm{d}T$, $T = \tau - t$. If the space-time is assumed static, $\dot{a} = 0$ and $G_{\text{effective}} = G$. In the dynamical case, Eq. (10) allows the appearance of a friction cosmological term in the Friedmann equation as follows [34]

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} + \underbrace{\frac{2(1-\alpha)}{T}\frac{\dot{a}}{a}}_{\bar{\Lambda}_{\text{dissipative}}/3}, \tag{12}$$

$$\frac{\ddot{a}}{a} + \frac{\alpha-1}{T}\frac{\dot{a}}{a} = -\frac{4\pi G\rho}{3} + \frac{\Lambda}{3}\,, \tag{13}$$

for zero pressure (dust), while in the case of radiation ($p = \rho/3$), we get the differential system

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} + \underbrace{\frac{2(1-\alpha)}{T}\frac{\dot{a}}{a}}_{\bar{\Lambda}_{\text{dissipative}}/3} \tag{14}$$

$$\frac{\ddot{a}}{a} + \frac{2(\alpha-1)}{T}\frac{\dot{a}}{a} = -\frac{8\pi G\rho}{3} + \frac{\Lambda}{3}\,, \tag{15}$$

or in the following forms

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} = \Lambda + \frac{2}{3}\bar{\Lambda}_{\text{dissipative}}\,, \qquad p = 0\,, \tag{16}$$

$$\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} = \frac{2}{3}(\Lambda + \bar{\Lambda}_{\text{dissipative}})\,, \qquad p = \frac{\rho}{3}\,. \tag{17}$$

The new dissipative cosmological constant term $\Lambda_{\text{dissipative}} \equiv 6(1-\alpha)\dot{a}/(aT)$ decays with time and vanishes as $T \to \infty$. This dissipative term can be introduced in the same way as in decaying two-fluid hydrodynamics and obeys the conservative law $T_{u\nu}^{\text{matter}} + T_{u\nu}^{\Lambda} + T_{u\nu}^{\text{curvature}} + T_{u\nu}^{\bar{\Lambda}} = 0$ where $T_{u\nu}^{\bar{\Lambda}}$ is the dissipative part of the cosmological constant and does not influence the matter-conservation law $T_{u\nu}^{\mathrm{matter}} = 0$. In the previous equation $T_{u\nu}^{\mathrm{curvature}} = -(1/(8\pi G))G_{u\nu}$ and $T_{u\nu}^{\Lambda} \equiv \Lambda/(8\pi G)$.

Motivated by the above investigations, we explore what follows some effects of the dissipative term in FRW cosmology. Note first that some results obtained from FRW standard cosmology are not consistent with recent observations [39–41]. Numerous models have been proposed to reconcile these contradictions with observation where few have given a satisfactory solution. The presence of a cosmological constant in a given cosmology prolongs the age of the universe. One of the motivations for introducing the dissipative lambda term is to reconcile the age parameter and the density parameter of the universe with current observational data. The inflationary paradigm requires the universe to have a critical density, but observations do not support this. In an attempt to resolve these problems with current observations, we suggest the following variation law for the cosmological constant $\Lambda = \widetilde{\alpha}\dot{a}^2/a^2 + \beta\ddot{a}/a + \gamma\dot{a}/(aT)$ and the cosmological matter density $\rho = (3\delta/(8\pi G))\dot{a}^2/a^2$, where $\widetilde{\alpha}$, β, γ and δ are constants [42–46]. Remember that the FRW Ricci scalar contains both terms $\dot{a}^2/a^2$ and $\ddot{a}/a$. Further, we assume (as in the literature) that the gravitational constant varies as $G = G_0(a/a_0)^m$, where m is a positive constant and $G_0(a_0)$ a is the present value of $G(a)$ [47–49 and references therein]. As a result, Eqs. (16) and (17) for matter and radiation dominated flat epoch yield

$$(1-\widetilde{\alpha})\frac{\dot{a}^2}{a^2} + (2-\beta)\frac{\ddot{a}}{a} - \frac{\gamma + 4(1-\alpha)}{T}\frac{\dot{a}}{a} = 0, \qquad p = 0 \tag{18}$$

$$\left(1 - \frac{2\widetilde{\alpha}}{3}\right)\frac{\dot{a}^2}{a^2} + \left(1 - \frac{2\beta}{3}\right)\frac{\ddot{a}}{a} - \frac{2}{3}\frac{\gamma + 6(1-\alpha)}{T}\frac{\dot{a}}{a} = 0, \qquad p = \frac{\rho}{3}\,, \tag{19}$$

We discuss in what follows the two cases.

Matter-dominated epoch

Different choices may hold. We are interested first on the simple choice $\gamma = 4(1-\alpha)$. The resulting general solution is then

$$a(T) = \left(C\frac{\beta + \widetilde{\alpha} - 3}{\beta - 2}T\right)^{(\beta-2)/(\beta+\widetilde{\alpha}-3)}, \tag{20}$$

where C is a constant, $\beta \neq 2$ and it corresponds to an accelerating expansion for $\widetilde{\alpha} < 1$. Note that if $\widetilde{\alpha} = 1$, $a \propto T^{(\beta-2)/(\beta-3)}$, while for $\beta = 0$, $a \propto T^{2/(\beta-3\widetilde{\alpha})}$, and it corresponds also to an accelerating cosmic expansion for $1 < \widetilde{\alpha} < 3$. We fall into the standard cosmology for $\widetilde{\alpha} = 0$ and $\beta = 0$, e.g., $a \propto T^{2/3}$. It follows then that

$$\Lambda = \left(\frac{\beta - 2}{\beta + \widetilde{\alpha} - 3}\right)\left(\frac{\beta - 2\widetilde{\alpha}}{\beta + \widetilde{\alpha} - 3} + 4(1-\alpha)\right)\frac{1}{T^2}\,, \tag{21}$$

$$\rho = \frac{3\delta R_0^m}{8\pi G_0} \frac{(\beta-2)^2 \Big(C(\beta+\widetilde{\alpha}-3)/(\beta-2) \Big)^{m(\beta-2)/(\beta+\widetilde{\alpha}-3)}}{(\beta+\widetilde{\alpha}-3)^2} \frac{1}{T^{(m(\beta-2)/(\beta+\widetilde{\alpha}-3))+2}}\,, \tag{22}$$

$$\bar{\Lambda}_{\text{dissipative}} = \frac{6(1-\widetilde{\alpha})(\beta-2)}{\beta+\widetilde{\alpha}-3} \frac{1}{T^2}\,, \tag{23}$$

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1-\widetilde{\alpha}}{2-\beta} \tag{24}$$

$$H \equiv \frac{\dot{a}}{a} = \frac{\beta-2}{\beta+\widetilde{\alpha}-3} \frac{1}{T}\,, \tag{25}$$

$$G \propto T^{m(\beta-2)/(\beta+\widetilde{\alpha}-3)}. \tag{26}$$

q and H are the deceleration and the Hubble parameters, respectively. Moreover, we need to have $\beta > 2$ for $q < 0$ and $HT > 1$, whereas G is increasing with the evolution of the universe for $m > 0$ and ensures that that energy density is a decreasing function of time. $\bar{\Lambda}_{\text{dissipative}} > 0$ for $\beta + \widetilde{\alpha} > 3$, while $\Lambda > 0$ for at least $\gamma > 0$ and $\beta > 2\widetilde{\alpha}$. The density parameter of the universe is $\Omega_M = \rho/\rho_c = \delta$ ($\rho_c = 3H^2/(8\pi G)$ is the critical density), the density parameter of the Λ-vacuum $\Omega_\Lambda = \Lambda/(3H^2) = \big(\beta - 2\widetilde{\alpha} + \gamma(\beta+\widetilde{\alpha}-3)\big)/[3(\beta-2)]$, whereas the density parameter of the $\bar{\Lambda}_{\text{dissipative}}$ is $\Omega(\bar{\Lambda}_{\text{dissipative}}) = \bar{\Lambda}_{\text{dissipative}}/(3H^2) = 2(1-\widetilde{\alpha})(\beta+\widetilde{\alpha}-3)/(\beta-2)$. It is easy to check that the total density parameter defined by $\Omega_T = \Omega_M + \Omega_\Lambda + \Omega_{\bar{\Lambda}}$ is positive and of about unity if $\beta - 2\widetilde{\alpha} + (\beta+\widetilde{\alpha}-3)\big(\gamma + 6(1-\widetilde{\alpha})\big) \approx 3(\beta-2)(1-\delta)$.

For $\delta \neq 4(1-\alpha)$, the dynamical equation is $(1-\widetilde{\alpha})T\dot{a}^2 + \big(4(\alpha-1)-\gamma\big)a\dot{a} = (\beta-2)Ta\ddot{a}$ with the general solution given by $a = A_1 \exp(B_1 T^w)$, $w = 2\big(4(\alpha-1)-\gamma\big)/(\beta-2)$ and $\beta + \widetilde{\alpha} = 3$. It corresponds to a fast inflation for $w \geq 1$. The arbitrary parameters A_1 and B_1 enter non-linearly in the solution. For $w = 1$, we recover the usual de-Sitter solution. In general, for $\widetilde{\alpha} = 0$, $\beta = 3$ and $\gamma = 4\alpha - 5$, the scale factor evolves as $a = A_1 \exp(B_1 T^2)$, corresponding to a super-inflationary scenario, and the vacuum cosmological constant evolves as $\Lambda = 3\ddot{a}/a + (4\alpha-5)\dot{a}/(aT)$, while the dissipative cosmological constant evolves as $\bar{\Lambda}_{\text{dissipative}} = 6(1-\alpha)\dot{a}/(aT)$. Finally, for the particular value $\beta = 2$, there exist non-static solution given by $a \propto T^{2(\gamma-4(\alpha-1))/(1-\widetilde{\alpha})}$, $\widetilde{\alpha} \neq 1$, and it corresponds for an accelerated expansion unless $2\big(\gamma-4(\alpha-1)\big) > (1-\widetilde{\alpha})$. This shows the importance of the dissipative cosmological constant or perturbed gravity that contributes to the accelerated expansion of the universe.

Radiation-dominated epoch

For the radiation-dominated universe, we will assume again the simple choice $\gamma = 6(1-\alpha)$. The resulting general solution is then

$$a(T) = \left(D\frac{2\beta + 2\widetilde{\alpha} - 6}{2\beta - 3}\right)^{(2\beta-3)/(2\beta+2\widetilde{\alpha}-6)}. \tag{27}$$

D is a constant, $\beta \neq 3/2$ and it corresponds to an accelerating expansion for $\widetilde{\alpha} < 3/2$. Note again that if $\widetilde{\alpha} = 1$, the scale factor evolves as $R \propto T$, while for $\beta = 0$, $a \propto T^{3/(6-2\widetilde{\alpha})}$ and corresponds to an accelerated expansion for $3/2 < \widetilde{\alpha} < 3$. We fall into the standard cosmology for $\widetilde{\alpha} = 0$ and $\beta = 0$, e.g., $a \propto T^{1/2}$. By comparing the scale evolution in the two epochs represented by Eqs. (20) and (27), it is not difficult to remark that it satisfies the observational requirement: the past transition of the universe from the acceleration to the deceleration expansion phase. It follows then that

$$\Lambda = \left(\frac{2\beta - 3}{2\beta + 2\widetilde{\alpha} - 6}\right)\left(\frac{2\beta - 3}{2\beta + 2\widetilde{\alpha} - 6} + 6(1-\alpha)\right)\frac{1}{T^2}, \tag{28}$$

$$\rho = \frac{3\delta R_0^m}{8\pi G_0}\,\frac{(2\beta-3)^2\left(C(2\beta+2\widetilde{\alpha}-6)/(2\beta-3)\right)^{m(\beta-3)/(\beta+\widetilde{\alpha}-3)}}{(2\beta+2\widetilde{\alpha}-6)^2}\,\frac{1}{T^{(m(2\beta-3)/(2\beta+2\widetilde{\alpha}-6))+2}}, \tag{29}$$

$$\bar{\Lambda}_{\text{dissipative}} = \frac{(3-2\widetilde{\alpha})(2\beta-3)}{\beta+\widetilde{\alpha}-3}\,\frac{1}{T^2}, \tag{30}$$

$$q = \frac{3 - 2\widetilde{\alpha}}{3 - 2\beta}, \tag{31}$$

$$H = \frac{2\beta - 3}{2\beta + 2\widetilde{\alpha} - 6}\,\frac{1}{T}, \tag{32}$$

$$G \propto T^{m(2\beta-3)/(2\beta+2\widetilde{\alpha}-6)}. \tag{33}$$

As a result, the density parameter of the universe in the radiation epoch $\Omega_\Lambda = \delta$ and the density parameter of the Λ-vacuum $\Omega_\Lambda = \big(2\beta+3+4\widetilde{\alpha}+\gamma(2\beta+2\widetilde{\alpha}-3)\big)/[3(2\beta-2)]$, whereas it is not difficult to check that the density parameter of the $\bar{\Lambda}_{\text{dissipative}}$-vacuum is $\Omega(\bar{\Lambda}_{\text{dissipative}}) = 2(3-2\widetilde{\alpha})(2\beta+2\widetilde{\alpha}-6)/[3(2\beta-3)]$. It is easy to check that the total density parameter is positive in the radiation-dominated epoch and of about unity if $2\beta + 3 - 4\widetilde{\alpha} + (2\beta + 2\widetilde{\alpha} - 6)(3\gamma + 6(3 - 2\widetilde{\alpha})) \approx 3(2\beta - 3)(1-\delta)$. Moreover, we need to have $\beta > 3/2$ for $q < 0$ and $HT > 1$, whereas G is increasing with the radiative-evolution of the universe for $m > 0$ and ensures that that energy density is a decreasing function of time. $\bar{\Lambda}_{\text{dissipative}} > 0$ for $\widetilde{\alpha} + \beta > 6$, while $\Lambda > 0$ for at least $\gamma > 0$ and $2\beta + 3 > 4\widetilde{\alpha}$.

More generally, for $\delta \neq 6(1-\alpha)$, the universe is governed in the radiation epoch by the dynamical equation $(3-2\widetilde{\alpha})T\dot{a}^2 + 2\big(6(\alpha-1)-\gamma\big)a\dot{a} = (2\beta-3)Ta\ddot{a}$ with the general solution given by $a = A_2\exp(B_2T^n)$, $n = 2\big(6(\alpha-1)-\gamma\big)/(2\beta-3)$ and $\beta+\widetilde{\alpha} = 3$. It corresponds to a fast inflation for $n \geq 1$. The arbitrary parameters A_2 and B_2 enter again non-linearly in the solution. For $n = 1$, we recover again the usual de-Sitter solution. Assuming $\widetilde{\alpha} = 1$, $\beta = 2$ and $\gamma = 6\alpha - 13/2$, the scale factor evolves again as $a = A_2\exp(B_2T^2)$, corresponding to a super-inflationary scenario where the vacuum and the dissipative cosmological constant evolve as $\Lambda = \dot{a}^2/a^2 + 2\ddot{a}/a + \big((12\alpha-13)/2\big)\dot{a}/(aT)$ and $\bar{\Lambda}_{\text{dissipative}} = 6(1-\alpha)\dot{a}/(aT)$ Note that in the matter-dominated epoch and within the same dynamics, Λ is independent of $\dot{a}^2/a^2$. For the particular value $\beta = 3/2$, the non-static solution is $a \propto T^{2(\gamma-6(\alpha-1))/(3-2\widetilde{\alpha})}$, $\widetilde{\alpha} \neq 3/2$, and it corresponds to an accelerated expansion unless $2\big(\gamma-6(\alpha-1)\big)/(3-2\widetilde{\alpha}) > 1$. An interesting feature arises also and corresponds to the particular value $\widetilde{\alpha} = 3/2$. For $a \neq 0$, the solution is simply given by $a \propto T^p$, with $p = 1+2\big((6(\alpha-1)-\gamma)/(2\beta-3)\big) > 1$ if $2\big((6(\alpha-1)-\gamma)/(2\beta-3)\big) > 1$. This proves again the importance of the dissipative cosmological constant, or perturbed gravity contributes to the accelerated expansion of the universe without invoking any exotic matter, phantom fields, quintessence, hessence, etc.

In conclusion, we have discussed a modified cosmological constant from fractional action-like variational approach with perturbed gravity, resulting in a dissipative cosmological constant. We have enlarged our solutions by introducing an increasing gravitational constant where we have proved their important role and contributions in describing the accelerated expansion of the big-bang universe. Our approach is an extension of the Einstein's general relativity. The age, horizon and flatness problems are better solved. It is, therefore, evident that FALVA cosmology with perturbed gravity, the presence of a second dissipative cosmological constant and the increasing gravitational constant enrich the big-bang cosmology. Finally, recent studies confirm the importance of fractional derivatives to portray more precisely the non-trivial behavior of FRW cosmology [50]. We would like to explore in a future work a generalized dissipative cosmology associated to a single-time action integral that can deal with different forms of friction. This could be done by working with fractional derivative as introduced in Ref. [50].

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RAZLOMNI DJELATNI VARIJACIJSKI PRISTUP, EINSTEINOVA GRAVITACIJA SA SMETNJOM I NOVA KOZMOLOGIJA

Razlomni račun je moćna metoda za nova istraživanja u klasičnoj fizici. Posebno je izgledno spajanje razlomnog diferencijalnog računa i kozmologije u traženju novih odnosa i odgovora na današnje probleme u astronomiji i fizici velikih sustava. U ovom se radu primjenjuje razlomni djelatni varijacijski pristup, koji je autor uveo 2005., za istraživanje nekih novih kozmoloških odnosa. Posebno se raspravlja promijenjena kozmološka konstanta na osnovi razlomnog djelatnog varijacijskog pristupa i Einsteinove gravitacije što vodi na disipativnu kozmološku konstantu.