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KANTOWSKI-SACHS COSMOLOGICAL MODELS WITH TIME-VARYING G AND Λ

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Kantowski-Sachs cosmological models with perfect fluid and time varying G and Λ are presented. Exact solutions of the field equations are obtained by using the scalar of expansion proportional to the shear scalar $\theta \propto \sigma$, which leads to a relation between the metric potentials $A = B^n$, where n is a constant. The corresponding physical interpretation of the cosmological solutions are also discussed.

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1. *Introduction*

In the Einstein theory of gravity, the Newtonian 'constant' G and cosmological 'constant' Λ are considered as fundamental constants. The gravitational 'constant' G plays the role of coupling constant between geometry of space and matter in the Einstein field equations. In an evolving universe, it appears natural to look at this constant as a function of time. Dirac [1] suggested a time-varying gravitational constant. The large-number hypothesis (LNH) proposed by Dirac [2, 3] leads to a cosmology where G varies with cosmic time. There have been many extensions of the Einstein's theory of gravitation, with time dependent G, in order to achieve a possible unification of gravitation and elementary-particle physics. Canuto and Narlikar [4] have shown that G varying cosmology is consistent with whatever cosmological observations are presently available. Researchers discussed the possibility of an increasing G [5, 6, 7].

The Λ term arises naturally in general relativistic quantum field theory where it is interpreted as the energy density of the vacuum (Zel'dovich [8, 9], Ginzburg et al. [10], Fulling et al. [11]). Cosmological models with variable G and Λ have been studied by a number of researchers [12 − 19] for an homogeneous and isotropic

FRW line element. Bianchi type-I models with variable G and Λ have been studied by Beesham [20], Vishwakarma et al. [21], Vishwakarma [22], Maharaj and Naidoo [23], Arbab [24, 25] and Pradhan and Yadav [26].

It is well known that exact solutions of general theory of relativity for homogeneous spacetimes belongs to either Bianchi type or to Kantowski-Sachs [27] type. Weber [28, 29] has done qualitative study of the Kantowski and Sachs [30] cosmological models. Lorcoz [31], Gron [32], Matravers [33], Krori et al. [34], Darowski [35] Li and Hao [36] have also studied cosmological models for the Kantowski-Sachs space-time. Singh and Agrawal [37] discussed Kantowski-Sachs type models in Saez and Ballester scalar tensor theory. Recently Pradhan and Yadav [26] have done Kantowski-Sachs model with variable G and Λ by assuming of a power-law time variation of the expansion factor.

In this paper, we consider space-time of Kantowski-Sachs model in a general form with variable G and Λ. We apply the equation of state $p = \omega\epsilon$ and scalar of expansion proportional to the shear scalar $\theta \propto \sigma$ [38].

2. *The Kantowski-Sachs model (check title)*

We consider the Kantowski-Sachs metric in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - A^2\mathrm{d}r^2 - B^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{1}$$

where A and B are functions of the cosmic time t alone.

Einstein's field equations with variable G and Λ in suitable units are

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G(t)T_{ij} + \Lambda(t)g_{ij}\,. \tag{2}$$

The energy momentum tensor for a perfect fluid is

$$T_{ij} = (\epsilon + p)v_i v_j - pg_{ij}\,, \tag{3}$$

where ϵ is the energy density of cosmic matter and p is its pressure.

Einstein's field equation (2) for the metric (1) leads to

$$2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} = 8\pi Gp - \Lambda\,, \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = 8\pi Gp - \Lambda\,, \tag{5}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{B^2} = -8\pi G\epsilon - \Lambda\,, \tag{6}$$

where the overhead dots denote the ordinary differentiation with respect to t.

In view of the vanishing divergence of the Einstein tensor, Eq. (2) gives

$$\dot{\epsilon} + (\epsilon + p)\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) + \epsilon\frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0\,. \tag{7}$$

We now assume that the law of conservation of energy ($T^{ij}_{;j}$) gives

$$\dot{\epsilon} + (\epsilon + p)\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = 0\,. \tag{8}$$

Using Eq. (7) yields

$$\dot{G} = -\frac{\dot{\Lambda}}{8\pi\epsilon} \tag{9}$$

indicating that G increases or decreases as Λ decreases or increases. We also consider the perfect fluid equation of state,

$$p = \omega\epsilon\,, \tag{10}$$

where ω, as suggested by Wang [39], may be defined as

$$\omega = \frac{1}{3}\frac{\epsilon_r}{(\epsilon_m + \epsilon_r)}\,, \tag{11}$$

with $\epsilon = \epsilon_m + \epsilon_r$, ϵ_m and ϵ_r being the matter crest mass and radiation energy densities. As the variation of $\omega(t)$ is slow as compared with the expansion of the universe, we expect that near the time when matter and radiation energy densities are equal, we can approximate $\omega(t)$ as the step function :

$$\omega \cong \begin{cases} 1/3 & \text{in the radiation dominated universe,} \\ 0 & \text{in the matter dominated universe.} \end{cases} \tag{12}$$

From Eqs. (4) and (5), we have

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}}{B}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{1}{B^2} = 0\,. \tag{13}$$

There are only five independent equations in six unknowns A, B, p, ϵ, G and Λ, so an extra equation is needed to solve the system completely. We assume the scalar of expansion to be proportional to the shear scalar, $\theta \propto \sigma$ [38], which leads to a relation between metric potentials

$$A = B^n, \tag{14}$$

where n is a constant.

Using Eqs. (14) and (13), we have

$$\frac{\ddot{B}}{B} + (n+1)\frac{\dot{B}^2}{B^2} + \frac{1}{1-n}\frac{1}{B^2} = 0, \quad n \neq 1\,. \tag{15}$$

Integrating Eq. (15), we obtain

$$\dot{B} = \frac{\sqrt{B^{2(n+1)} + C_1(n^2-1)}}{B^{(n+1)}\sqrt{n^2-1}}\,, \tag{16}$$

where C_1 is a constant of integration. With the help of Eqs. (14) and (16), the line element (1) reduces to

$$\mathrm{d}s^2 = \frac{B^{2(n+1)}(n^2-1)}{B^{2(n+1)} + C_1(n^2-1)}\mathrm{d}B^2 - B^{2n}\mathrm{d}r^2 - B^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{17}$$

By suitable transformation of coordinates, the line element (17) reduces to

$$\mathrm{d}s^2 = (n^2-1)\frac{T^{2(n+1)}}{T^{2(n+2)} + C_1(n^2-1)}\mathrm{d}T^2 - T^{2n}\mathrm{d}r^2 - T^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{18}$$

For the model (18), the geometrical and physical parameters can be easily obtained. The expressions for the energy density ϵ, gravitational constant $G(t)$ and cosmological constant $\Lambda(t)$ are given by

$$\epsilon = \frac{k}{T^{(n+2)(\omega+1)}}\,, \quad k = constant \tag{19}$$

$$G(t) = \frac{T^{(n+2)(\omega+1)}}{8\pi k(\omega+1)}\left[\frac{2n}{(1-n^2)T^2} - \frac{2(2n+1)C_1}{T^{2n+4}}\right] \tag{20}$$

$$\Lambda(t) = \frac{(2n+1)C_1}{T^{2n+4}}\left(\frac{1-\omega}{1+\omega}\right) - \frac{n^2(\omega+1)+2n\omega}{(n^2-1)(\omega+1)T^2}. \tag{21}$$

The expansion scalar θ and scalar σ for the model (18) are

$$\theta = \frac{(n+2)}{\sqrt{n^2-1}}\frac{\sqrt{T^{2n+2} + C_1(n^2-1)}}{T^{n+2}} \tag{22}$$

$$\sigma = \frac{(n-1)}{\sqrt{3}(n^2-1)}\frac{\sqrt{T^{2n+2} + C_1(n^2-1)}}{T^{n+2}} \tag{23}$$

For $\epsilon > 0$, we require $k > 0$. The model has singularity at $T = 0$. The model starts with ϵ, θ, σ and Λ, all being infinite and continues to expand till $T = \infty$. For this model, the scale factors are zero at $T = 0$, which shows that the spacetime exhibits point type singularity. Gravitational constant $G(t)$ is zero initially and gradually increases and tends to infinity at late times. Since $\sigma/\theta = constant$, the model does not approach isotropy for large value of T. Therefore, the model describes a continuously expanding, shearing, non-rotating universe with the big-bang start. In this model we observe that the cosmological term Λ is infinite initially, gradually decreases and becomes zero at late times.

In the special case of $C_1 = 0$, from Eq. (16) the line element (1) reduces to

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{t^{2n}}{(n^2-1)^n}\mathrm{d}r^2 - \frac{t^2}{(n^2-1)}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{24}$$

After suitable transformation, Eq. (24) reduces to

$$\mathrm{d}s^2 = \mathrm{d}T^2 - T^{2n}\mathrm{d}r^2 - T^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{25}$$

The physical and geometrical parameters of the model (25) are

$$\epsilon = \frac{k_2}{T^{(n+2)(\omega+1)}}\,, \qquad k_2 = constant\,, \tag{26}$$

$$G(t) = \frac{-n}{4\pi(\omega+1)k_2}T^{(n+2)\omega+n}\,, \tag{27}$$

$$\Lambda(t) = \frac{-n(n+2\omega+n\omega)}{(\omega+1)T^2}\,. \tag{28}$$

The shear σ and expansion scalar θ are given by

$$\sigma = \frac{(n-1)}{\sqrt{3}\,T}\,, \tag{29}$$

$$\theta = \frac{(n+2)}{T}\,. \tag{30}$$

Since $\sigma/\theta = constant$, the model does not approach isotropy. We can obtain the deceleration parameter $q = (1-n)/(n+2)$, which shows that the deceleration parameter is constant. The model of constant deceleration parameter has been considered by Berman and Som [40]. The Hubble parameter H reads as

$$H = \frac{n+2}{3T}\,, \tag{31}$$

which can be written as

$$H = \frac{n-1}{3qT} \,. \tag{32}$$

For the present phase p, we obtain

$$T_p = \frac{1-n}{3q_p H_p} \,. \tag{33}$$

It is evident that negative q_p would increase the present age of the universe. From Eq. (27), we obtain

$$\frac{\dot{G}}{G} = \frac{(n+2)\omega + n}{T} \,, \tag{34}$$

and the present value is

$$\left(\frac{\dot{G}}{G}\right)_p = \frac{3[(n+2)\omega + n]}{1-n} \, H_p q_p \,. \tag{35}$$

We can find that the quantity $G\epsilon$ satisfies the condition for a Machian cosmological solution, i.e. $G\epsilon \sim H^2$, which follows from the model of Kalligas et al. [15].

For the energy density to be positive definite, we must have $k_2 > 0$. The energy density decreases as time increases and tends to zero as T tends to infinity. We also observe that the spatial volume is zero at $T = 0$. Thus, the singularity exists at $T = 0$ in the model. The gravitational constant is zero initially and gradually decreases and tends to infinity at late times, whereas cosmological term $\Lambda(t)$ varies as square of the age of universe and tends to zero as $T \to \infty$. It is noted that negative value of G and Λ had already been considered by Vishwakarma [22] and Singh et al. [18]. The deceleration parameter is constant for all time. For $n = 1$, $T_0 H_0 = 1$. This is within the current limits for the universe age $0.8 < T_0 H_0 < 1.3$ and is in good agreement with the best observation $T_0 H_0 \sim 1$ [41].

In the model (25) we observe that $\Lambda \sim H^2$ and $\Lambda \sim T^{-2}$, which is in accordance with the main dynamical laws one finds in the literature proposed for the decay of Λ. The dynamical law $\Lambda \sim H^2$ has be proposed by Carvalho et al. [42] and considered by Salim and Waga [43], Arbab and Abdel-Rahman [44], Wetterich [45] and Arbab [46]. In view of the present estimates, Λ is of the order H_0^2 [47]. The dynamical law $\Lambda \sim T^{-2}$ has been considered by several authors, e.g. Lau [48], Berman [14], Beesham [20], Lopez et al. [49], Bertolami [50], Endo et al. [51] and Canuto et al. [52], to mention a few.

3. *Conclusion*

We have obtained exact solutions of the field equations for Kantwoski-Sach space-time with variable gravitational constant $G(t)$ and cosmological constant $\Lambda(t)$

in the presence of a perfect fluid. In general, the space time exhibits "point type singularity" at initial stage and gravitational constant is zero, but cosmological term varies as square of the age of universe. In the model, cosmological term Λ is infinite at the beginning and it decreases to become zero at late times. Deceleration parameter is constant for all time. In the model we obtain $\Lambda \sim H^2$ and $\Lambda \sim T^{-2}$ which is in accordance with the main dynamical laws for the decay of Λ. We also obtain that the model satisfies the condition for a Machian cosmological solution, i.e. $G\epsilon \sim H^2$, which follows from the model of Kalligas et al. [15].

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KANTOWSKI-SACHSOVI KOZMOLOŠKI MODELI S VREMENSKI-PROMJENLJIVIM G I Λ

Predstavljamo Kantowski-Sachsove kozmološke modele s vremenski-promjenljivim G i Λ koji sadrže perfektnu tekućinu. Izvodimo egzaktna rješenja jednadžbi polja primjenom skalara razvoja koji je razmjeran posmičnom skalaru, $\theta \propto \sigma$, što vodi na relaciju među metričkim potencijalima $A = B^n$, gdje je n stalan. Raspravljamo i fizičko tumačenje kozmoloških rješenja.