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PATH INTEGRAL FORMULATION OF SIGMA MODEL WITH NONCOMMUTATIVE FIELD SPACE

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We reduce the sigma model with noncommutative field space to the quantum mechanical problem of two-dimensional harmonic oscillators with noncommutating coordinates. Then, by means of the Lagrangian formulation of the noncommutative quantum mechanics, we re-derive the finite-temperature partition function of the system derived earlier in the context of the Hamiltonian formulation of the model.

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1. *Introduction*

Most of the recent studies in noncommutative (NC) field theories [1 − 4] consider the NC space-time and assume the field space of the underlying theories to be commutative. As a result, the noncommutativity reveals itself through the interaction vertices and thus via the interactions. So, for free non-interacting theories there are no modifications arising from the noncommutativity of space-time. However, the situation changes when the noncommutativity is implemented in target space manifold rather than the space-time manifold. Therefore, in contrast to the case of NC space-time manifold, the free part of the action modifies when the NC target space manifold is considered. The other characteristics of the theories with noncommutative field space is their violation from the Lorentz invariance [5, 6]. In Ref. [7] authors have studied the black-body spectrum of a sigma model with NC field space. The possible generalizations to the U(1) gauge field is considered in Ref. [8].

In this short paper we shall consider a 1+1 dimensional sigma model with NC field space and shall apply the path integral approach to the NC quantum mechanics developed earlier in Refs. [9 − 12] to derive the finite-temperature partition function of the model. Our result is in accordance with the expression derived for the partition function by applying the method based on the Hamiltonian formalism [7]. As is expected, in the $\theta \to 0$ limit, we recover the partition function of the model with usual commutative field space.

2. *Two-dimensional sigma model*

We consider a sigma model with classical action in 1+1 dimension as

$$S = g \int \mathrm{d}^2x \, \partial_\alpha \phi^A \, \partial^\alpha \phi^A, \tag{1}$$

with $\alpha, A = 1, 2$ and g as the coupling constant. The corresponding Lagrangian reads

$$L = g \int \mathrm{d}x \big(\dot{\phi}^A \dot{\phi}^A - \phi_x^A \phi_x^A \big) \, . \tag{2}$$

with $\phi_x^A \equiv \partial_x \phi^A$ and $\dot{\phi}^A \equiv \partial_t \phi^A$. The fields ϕ^A and their conjugate momenta π^A fulfil the equal-time canonical structure (setting $\hbar = 1$)

$$\Big[\phi^A(t,x), \pi^B(t,x') \Big] = \mathrm{i}\delta^{AB} \delta(x - x') \, , \tag{3}$$

$$\Big[\phi^A(t,x), \phi^B(t,x') \Big] = 0 \, ,$$

$$\Big[\pi^A(t,x), \pi^B(t,x') \Big] = 0 \, .$$

For the fields defined over the length $l(= 1)$, we can expand them in terms of the normal modes $\chi_n^A(t)$ as

$$\phi^A(t,x) = \sum_{n=1}^{\infty} \chi_n^A(t) \sin(\pi n x) \, , \tag{4}$$

with immediate result for the Lagrangian and Hamiltonian as

$$L = \frac{g}{2} \sum_n \Big(\dot{\chi}_n^A \dot{\chi}_n^A - \omega_n^2 \chi_n^A \chi_n^A \Big) \tag{5}$$

and

$$H = \frac{1}{2g} \sum_n \Big(p_n^A p_n^A + g^2 \omega_n^2 \chi_n^A \chi_n^A \Big), \tag{6}$$

where $\omega_n = \pi n$. Now, the expressions (5) and (6) can be interpreted as the Lagrangian and Hamiltonian of an infinite series of uncoupled harmonic oscillators with mass g.

3. *Noncommutative field space*

Noncommutative deformation of the field space could be implemented via the deformation of canonical structure to a noncommutative one

$$\left[\widehat{\phi}^A(t,x), \widehat{\pi}^B(t,x')\right] = \mathrm{i}\delta^{AB}\delta(x-x')\,, \tag{7}$$

$$\left[\widehat{\phi}^A(t,x), \widehat{\phi}^B(t,x')\right] = \mathrm{i}\theta\varepsilon^{AB}\delta(x-x')\,,$$

$$\left[\widehat{\pi}^A(t,x), \widehat{\pi}^B(t,x')\right] = 0\,,$$

which leads to the corresponding structure for the expansion modes

$$\left[\widehat{\chi}_n^A(t), \widehat{p}_m^B(t)\right] = \mathrm{i}\delta^{AB}\delta_{n,m}\,, \tag{8}$$

$$\left[\widehat{\chi}_n^A(t), \widehat{\chi}_m^B(t)\right] = \mathrm{i}\theta\varepsilon^{AB}\delta_{n,m}\,,$$

$$\left[\widehat{p}_n^A(t), \widehat{p}_m^B(t)\right] = 0\,.$$

For the anti-symmetric matrix ε^{AB} we assume $\varepsilon^{12} = 1$.

However, one recovers the usual canonical structure (3) if we redefine the noncommutative fields $\widehat{\phi}^A(x,t)$ in terms of the commutative fields $\phi^A(x,t)$ via

$$\widehat{\phi}^A(x,t) = \phi^A(x,t) - \frac{\theta}{2}\varepsilon^{AB}\widehat{\pi}^B(x,t)\,, \tag{9}$$

$$\widehat{\pi}^A(x,t) = \pi^A(x,t)\,,$$

and

$$\widehat{\chi}_n^A(t) = \chi_n^B(t) - \frac{\theta}{2}\varepsilon^{AB}p_n^B(t)\,, \tag{10}$$

$$\widehat{p}_n^A(t) = p_n^A(t)\,.$$

Therefore, upon substituting for the noncommutative fields from (10) in (5) and (6), we find the θ-deformed Hamiltonian and Lagrangian [9−11]

$$H_\theta = \sum_n \frac{\kappa_n}{2g} p_n^A p_n^A + \frac{g}{2}\omega_n^2 \chi_n^A \chi_n^A - \frac{g}{2}\theta\omega_n^2 \varepsilon^{AB}\chi_n^A p_n^B , \tag{11}$$

and

$$L_\theta = \sum_n \frac{g}{2\kappa_n}\dot\chi_n^A \dot\chi_n^A - \frac{g\omega_n^2}{2\kappa_n}\chi_n^A\chi_n^A + \frac{g^2\omega_n^2\theta}{2\kappa_n}\varepsilon^{AB}\chi_n^A\dot\chi_n^B \tag{12}$$

with the θ-deformed action as

$$S_\theta\big[\phi^1,\phi^2\big] = \frac{g}{2}\int \mathrm{d}t \sum_n \left(\frac{1}{\kappa_n}\dot\chi_n^A\dot\chi_n^A - \frac{\omega_n^2}{\kappa_n}\chi_n^A\chi_n^A + \frac{g\omega_n^2\theta}{\kappa_n}\varepsilon^{AB}\chi_n^A\dot\chi_n^B\right) \tag{13}$$

$$= \sum_n \bar S_\theta\big[\chi_n^1,\chi_n^2\big],$$

where $\kappa_n = 1 + \frac{1}{4}g^2\theta^2\omega_n^2$.

4. *Finite-temperature partition function*

For the Euclidean-time action, the finite-temperature partition function is

$$Z_\theta(\beta) = \int D\phi^1 D\phi^2 \exp\big[-S_\theta^E[\phi^1,\phi^2]\big] = \int \prod_n D\chi_n^1 D\chi_n^2 \exp\Big[-\sum_n \bar S_\theta^E[\chi_n^1,\chi_n^2]\Big]$$

$$= \prod_n \mathrm{Tr}\, K\big(\vec\chi_n'', \beta; \vec\chi_n', 0\big)\,, \tag{14}$$

with $\vec\chi_n = \big(\chi^1,\chi^2\big)$ and

$$\bar S_\theta^E[\chi_n^1,\chi_n^2] = \frac{g}{2}\int_0^\beta \mathrm{d}\tau \left(\frac{1}{\kappa_n}\dot\chi_n^A\dot\chi_n^A + \frac{\omega_n^2}{\kappa_n}\chi_n^A\chi_n^A + \frac{g\omega_n^2}{\mathrm{i}\kappa_n}\theta\varepsilon^{AB}\chi_n^A\dot\chi_n^B\right) \tag{15}$$

$$= \frac{g\omega_n}{2\sqrt{\kappa_n}\sinh\big(\beta\omega_n\sqrt{\kappa_n}\big)}\Bigg[\big(\vec\chi_n''^2 + \vec\chi_n'^2\big)\cosh\big(\beta\omega_n\sqrt{\kappa_n}\big)$$

$$-2\big(\vec\chi_n'\cdot\vec\chi_n''\big)\cosh\big(\beta\omega_n\sqrt{\kappa_n-1}\big) + 2\big(\vec\chi_n'\times\vec\chi_n''\big)_z \sinh\big(\beta\omega_n\sqrt{\kappa_n-1}\big)\Bigg]$$

$$\equiv A\big(\vec{\chi}''_n, \beta; \vec{\chi}'_n, 0\big)\,,$$

where $\vec{\chi}''_n = \vec{\chi}(\beta)$ and $\vec{\chi}'_n = \vec{\chi}_n(0)$. Here the dots stand for the derivative with respect to τ $(= \mathrm{i}t)$. The single-particle θ-deformed propagator is [9−12]

$$K_\theta\big(\vec{\chi}''_n, \beta; \vec{\chi}'_n, 0\big) = \frac{g\omega_n}{2\pi\sqrt{\kappa_n}\sinh\big(\beta\omega_n\sqrt{\kappa_n}\big)} \exp\Big(-A\big(\vec{\chi}''_n, \beta; \vec{\chi}'_n, 0\big)\Big). \tag{16}$$

For the trace of the exponential term in (16) we find

$$\mathrm{Tr}\ \mathrm{e}^{-A} = \frac{\pi\sqrt{\kappa_n}}{g\omega_n}\ \frac{\sinh\big(\beta\omega_n\sqrt{\kappa_n}\big)}{\cosh\big(\beta\omega_n\sqrt{\kappa_n}\big) - \cosh\big(\beta\omega_n\sqrt{\kappa_n - 1}\big)}\,. \tag{17}$$

Therefore, one is left with the partition function

$$Z_\theta(\beta) = \prod_n \frac{g\omega_n}{2\pi\sqrt{\kappa_n}\sinh\big(\beta\omega_n\sqrt{\kappa_n}\big)} \mathrm{Tr}\ \mathrm{e}^{-A} \tag{18}$$

$$= \prod_n \frac{1}{4\sinh\big(\beta\omega_n\gamma_{+,n}\big)\sinh\big(\beta\omega_n\gamma_{-,n}\big)}$$

$$= \exp\Big(-\beta\sum_n \omega_n\sqrt{\kappa_n}\Big)\prod_n \frac{1}{\big(1-\exp\big(-2\beta\omega_n\gamma_{+,n}\big)\big)\big(1-\exp\big(-2\beta\omega_n\gamma_{-,n}\big)\big)}\,,$$

where $\gamma_{\pm,n} = \frac{1}{2}(\sqrt{\kappa_n} \pm \sqrt{\kappa_n - 1})$. Hence, by excluding the divergent contribution, $\sum_n \omega_n\sqrt{\kappa_n}$, one finds the free energy

$$F_\beta(\beta) = -\frac{1}{\beta}\ \ln Z_\theta(\beta) \tag{19}$$

$$= \sum_n \Big[\ln\big(1-\exp(-2\beta\omega_n\gamma_{+,n})\big) + \ln\big(1-\exp(-2\beta\omega_n\gamma_{-,n})\big)\Big],$$

which is the result derived earlier by employing the Hamiltonian formalism [7]. In the $\theta \to 0$ limit, $\gamma_{\pm,n}$ tends to $\frac{1}{2}$, and one recovers the usual free energy

$$F(\beta) = 2\sum_n \ln\big(1-\exp(-\beta\omega_n)\big)\,. \tag{20}$$

The θ-deformed partition function, up to the first order in noncommutativity parameter, reads

$$Z_\theta(\beta)=\exp\left(\frac{\beta\pi}{12}\right)\prod_n \frac{1}{\big[1-\exp\big(-\beta\omega_n(1+\frac{1}{2}g\theta\omega_n)\big)\big]\big[1-\exp\big(-\beta\omega_n(1-\frac{1}{2}g\theta\omega_n)\big)\big]} \tag{21}$$

where we have invoked the Riemann zeta function $\zeta(-1)$ regularization with

$$\zeta(-1) = -\frac{1}{12}. \tag{22}$$

5. *Conclusion*

By expanding the fields of a sigma model with noncommutative field space in terms of the Fourier modes, we found its classical action in terms of the infinite series of actions governing the dynamics of two dimensional non-commutative harmonic oscillators. Then we derived the partition function of the model by calculating the trace of propagators associated with oscillators. In the limit $\theta \to 0$, the familiar result of model with commutative field space is recovered.

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IZLAGANJE SIGMA MODELA INTEGRALIMA PO PUTU U NEKOMUTATIVNOM PROSTORU POLJA

Sažimamo sigma model u nekomutativnom prostoru polja na kvantno-mehanički problem dvodimenzijskog harmoničkog oscilatora s nekomutativnim koordinatama. Zatim, pomoću prikaza nekomutativne kvantne mehanike preko lagrangiana, izvodimo particijsku funkciju sustava na konačnoj temperaturi koja je ranije izvedena u okviru prikaza modela preko hamiltonijana.