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GENERATING FUNCTIONAL FOR BOUND STATES IN QED

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The manifestly Lorentz covariant formulation of quantum electrodynamics disregards Coulomb instantaneous interaction and its consequence – instantaneous bound states (IBS's). In this article we consider the way of construction the IBS generating functional using the operator generalization of the initial data in the Dirac Hamiltonian approach to QED.

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1. Introduction

The first papers by Dirac [2], Heisenberg, Pauli [9], and Fermi [7] on the quantization of electrodynamics ran into difficulties with the determination of physical variables. The interpretation of all gauge components as independent variables contradicts the quantum principles, whereas excluding nonphysical variables contradicts the relativistic principles.

The first quantization of electrodynamics was made by Dirac [2] who disregarded the relativistic principles and excluded nonphysical components by the reduction of the initial action to the solution of the Gauss law constraint, i.e. the equation for the time-like component of the gauge vector field. The Gauss law connects initial data of the time-like component with the data of all other fields. But this *constraint-shell* method had a set of defects, including nonlocality, explicit noncovariance as the dependence on the external time axis of quantization, and complexity. Feynman-Schwinger-Tomonaga formulation of QED admitted a simpler method based on the extended dynamics where all components were considered on equal footing with fixing a relativistic gauge.

At the beginning of the sixties of the twentieth century, Feynman found that the naive generalization of his method of construction of QED did not work for the non-Abelian theory. The unitary S-matrix in the non-Abelian theory was obtained in the form of the Faddeev-Popov (FP) path integral [6] by the brilliant application of the theory of connections in vector bundle. There is an opinion that the FP path integral is the highest level of quantum description of gauge relativistic constrained systems. In any case, just this FP integral was the basis to prove renormalizability of the unified theory of electroweak interactions in papers by 't Hooft and Veltman, awarded the Nobel prize in 1999.

Nevertheless, in the context of the first Dirac quantization and its Hamiltonian generalizations [3, 8], the intuitive status of the FP integral was so evident that two years after the paper [6], Faddeev gave the foundation of the FP integral by the construction of the unitary S-matrix [5] for an "equivalent non-Abelian unconstrained system", derived by resolving constraints in terms of the *radiation variables* of the Hamiltonian description.

Faddeev showed that on the one hand, the constraint-shell dynamics is compatible with the simplest quantization by the standard Feynman path integral, on the other hand, this Feynman integral is equivalent to the FP integral in an arbitrary gauge. This equivalence was proved by the change of variables in the Feynman path integral that removed the time-like vector of the canonical quantization into the phase factors of physical source terms. These phase factors disappear for S-matrix elements on the mass shell of elementary particles. In other words, Faddeev proved the equivalence of the constraint-shell approach with quantization of gauge theories by the gauge-fixing method only for scattering amplitudes [5] where all color particle-like excitations of the fields are on their mass-shell. But the scattering amplitudes for color particles are nonobservable in QCD. The observables are hadrons as colorless bound states where elementary particles are off mass-shell. Just for this case, the Faddeev theorem of equivalence of different gauges becomes problematic even for QED in the sector of instantaneous bound states, as the FP integral in a relativistic gauge loses all propagators with analytic properties that lead to instantaneous bound states identified with observable atoms.

It is not known how to generalize Faddeev equivalence theorem onto bound states even for quantum electrodynamics. In this article, we present a construction of a generating functional for bound states in QED using another possibility. We generalize the Dirac constraint-shell approach to gauge theories. It seems that this method can lead to the relativistic covariant unitary renormalizable scattering theory of instantaneous bound states in QED. An extension of this framework to non-Abelian gauge theories paves a way for a description of hadronization and confinement in QCD. This is our main motivation for this line of research.

2. Bound state S-matrix in QED: statement of the problem

To state the problem, let us recall the Dirac approach [2] to QED and its relation to the standard Faddeev-Popov [6] path integral quantization of gauge theories. Our starting point is the well known QED action

$$W[A,\psi,\bar\psi] = \int \mathrm{d}^4x \Big\{ -\frac{1}{4}\,[\partial^\mu A^\nu - \partial^\nu A^\mu]^2 + \bar\psi[\mathrm{i}\gamma^\mu\partial_\mu - m]\psi + A_\mu j^{\,\mu} \Big\}, \qquad (1)$$

where A_μ is the vector potential, ψ is the Dirac electron-positron bispinor field and $j_\mu = e\bar\psi\gamma_\mu\psi$ is the charge current. Dirac [2] proposed to eliminate the time component of the four-vector potential by the substitution of the manifest resolution of the Gauss constraint $\delta W/\delta A_0 = 0$ into the initial action (1) in the rest frame $\ell^{(0)}_\mu = (1,0,0,0)$. A result of such reduction procedure written in terms of the so called radiation variables[1] $A^*_a = (A^*_1, A^*_2), \psi^*, \bar\psi^*$ leads to the following constraint-shell action

$$\begin{aligned} W^*[A^*,\psi^*,\bar\psi^*] = {} & W\Big|_{\delta W/\delta A_0 = 0} \\ = {} & \int \mathrm{d}^4x \frac{1}{2}\left[\sum_{a=1,2} (\partial_\mu A^*_a \partial^\mu A^*_a) + \frac{1}{2} j^*_0 \frac{1}{\Delta} j^*_0 - j^*_k A^*_k + \bar\psi^*(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi^*\right], \end{aligned} \qquad (2)$$

where $A^*_k = \sum\limits_{a=1,2} A^*_a e^a_k, \quad k=1,2,3$ and $\frac{1}{\Delta} j^*_0(\mathbf{x},\mathbf{t}) \overset{\mathrm{def}}{=} -\frac{\mathbf{1}}{\mathbf{4}\pi}\int \mathrm{d}^\mathbf{3}\mathbf{y}\,\frac{\mathbf{j^*_0}(\mathbf{y},\mathbf{t})}{|\,\mathbf{x}-\mathbf{y}|}$.

Having (2), one can define the constraint-shell path integral

$$Z^*[s^*,\bar s^*, J^*|\ell^{(0)}] = \int \prod_{a=1,2} DA^*_a D\psi^* D\bar\psi^* \mathrm{e}^{\mathrm{i}W^*[A^*,\psi^*,\bar\psi^*] + \mathrm{i}S^*}, \qquad (3)$$

where

$$S^* = \int \mathrm{d}^4x \left(\bar s^*\psi^* + \bar\psi^* s^* + \sum_{a=1,2} J^*_a A^*_a\right) \qquad (4)$$

is the external source term. The theory defined by (3) is frame-depended. This frame dependence can be removed by two steps [5]. First, we make the identical change of radiation variables by Lorenz-type "dummy" variables

$$A^*_k[A^F] = A^F_k - \partial_k \Lambda(A^F), \qquad \psi^*[A^F] = \exp\left\{\mathrm{i}e\Lambda(A^F)\right\}\psi\,, \qquad (5)$$

where $\Lambda(A^F) = \Delta^{-1}\,\partial^j A^F_j$. Then the constraint-shell generating functional can be rewritten in the following form

$$\begin{aligned} Z^*[s^*,\bar s^*,J^*|\ell^{\,(0)}] = {} & \int \prod_\mu DA^F_\mu D\psi^F D\bar\psi^F \Delta^F_{FP}\delta(F(A^F)) \times \\ & \times\ \mathrm{e}^{\mathrm{i}W[A^F,\psi^F,\bar\psi^F] + S^*[\ell^{(0)}]}. \end{aligned} \qquad (6)$$

[1] Dirac's radiation variables are gauge-invariant functionals of the initial fields A_μ, ψ, $\bar\psi$ describing physical degrees of freedom. How to construct these observables, see Refs. [15, 18].

The change of variables (5) introduces the additional nonphysical degrees of freedom and the Faddeev-Popov determinant of the transition to new variables of integration. The nonphysical degrees of freedom are removed by the delta function gauge condition. However, the dependence on the time axis is still present in the source term. To remove the frame dependence completely, we make the change of sources

$$S^*[\,\ell\,] \Rrightarrow S^F, \qquad S^F = \int \mathrm{d}^4x \left(\bar{s}^F\psi^F + \bar{\psi}^F s^F + A^F_\mu J^\mu\right). \tag{7}$$

After that the constraint-shell generating functional $Z^*[s^*, \bar{s}^*, J^*|\ell^{(0)}]$ takes the equivalent form of the Faddeev-Popov integral

$$\begin{aligned} Z^F[s^F, \bar{s}^F, J^F] &= \int \prod_\mu DA^F_\mu D\psi^F D\bar{\psi}^F \Delta^F_{FP}\delta(F(A^F)) \times \\ &\times \quad \mathrm{e}^{\mathrm{i}W[A^F, \psi^F, \bar{\psi}^F] + S^F}. \end{aligned} \tag{8}$$

Thus the fundamental constraint-shell generating functional (3) coincides with the Faddeev-Popov integral (8), if the change of sources (7) is valid. However, the validity of this change was shown only in the sector of scattering amplitudes derived from Green functions for elementary particles on their mass-shell [5].

On the other side, it is not known how to generalize the Faddeev-Popov path integral approach onto instantaneous bound states formed by the Coulomb singularity. Moreover, nobody proved that quantum electrodynamics based on Z^F contains instantaneous (observed) bound states. Really, the Faddeev-Popov perturbation theory in the relativistic gauge (8) contains only photon propagators with the light-cone singularities forming the Wick-Cutkosky bound states [10] with the spectrum different from the observed one. These bound states have the problem of tachyons and the probability interpretation.

Therefore the question appears, what is the Faddeev equivalence theorem for instantaneous bound states, where elementary particles are off mass-shell?

Our proposition is based on the generalization [16] of the above described formulation, where the parameters of the comoving frame of reference in the Dirac approach to QED become eigenvalues of the total momentum operator of instantaneous bound states

$$\ell^{(0)}_\mu \to \ell_\mu \to \hat{\ell}_\mu \sim \frac{\partial}{\partial X^\mu}\,, \qquad X^\mu = \sum_{J=1}^{N} x^\mu_J\,, \tag{9}$$

so that these eigenvalues are proportional to total momenta of bound states

$$\hat{\ell}^\mu \; |\Phi_B(P_B)\rangle = \frac{P^\mu_B}{M_B}\; |\Phi_B(P_B)\rangle\,, \qquad P^\mu P_\mu = M^2_B\,. \tag{10}$$

This framework yields the observed spectrum of bound states which corresponds to the instantaneous Coulomb interaction and paves a way for constructing an instantaneous bound state (IBS) generating functional

$$Z^*[\,\ell^{(0)}\,] \to Z^*[\,\ell\,] \to Z^*_{\mathsf{IBS}}[\,\hat{\ell}\,]\,, \tag{11}$$

and IBS S-matrix.

3. Bound state generating functional in QED

The instantaneous Coulomb interaction is described in (2) by the zero component current-current term, which can be converted into the following form

$$\begin{aligned} W^*_{\mathrm{Coulomb}}[\ell^{(0)}] &= \frac{1}{2}\int \mathrm{d}^4x_1\mathrm{d}^4x_2\,\psi(x_2)\bar{\psi}(x_1)\mathcal{K}(x_1,x_2|\ell^{(0)})\psi(x_1)\bar{\psi}(x_2) \\ &\equiv \frac{1}{2}\left(\psi_2\bar{\psi}_1,\mathcal{K}_{(1,2)}(\ell^{(0)})\psi_1\bar{\psi}_2\right), \end{aligned} \tag{12}$$

where

$$\mathcal{K}(x_1,x_2|\ell^{(0)}) = \ell^{(0)\nu}\gamma_\nu V(|\,z^\perp_\mu(\ell^{(0)})|)\ell^{(0)\nu}\gamma_\nu\delta(z_\mu\ell^{(0)\mu})$$

$$= \gamma_0 V(|\,\mathbf{x_1}-\mathbf{x_2}|)\gamma_\mathbf{0}\delta(\mathbf{x_{(1)0}}-\mathbf{x_{(2)0}})$$

is the Coulomb kernel written in the rest frame $\ell^{(0)}_\mu = (1,0,0,0)$. The Coulomb kernel contains the Coulomb potential $V(|\,z^\perp_\mu(\ell^{(0)})|) = -e^2\left[4\pi|\,z^\perp_\mu(\ell^{(0)})|\right]^{-1} = -e^2\left[4\pi|\,\mathbf{x_1}-\mathbf{x_2}|\right]^{-1}$ depending on the transverse components $z^\perp_\mu(\ell^{(0)}) = z_\mu - \ell^{(0)}_\mu(z_\nu\ell^{(0)\nu})$ of the relative coordinate $z_\mu = x_{(1)\mu} - x_{(2)\mu}$. Since the theory is Lorentz covariant [20, 18, 15], the Coulomb kernel can be transformed to an arbitrary frame $\mathcal{K}(x_1,x_2|\ell^{(0)}) \Rrightarrow \mathcal{K}(x_1,x_2|\ell) = \ell^\nu\gamma_\nu V(|\,z^\perp_\mu(\ell)|)\ell^\nu\gamma_\nu\delta(z_\mu\ell^\mu)$. Such form of $\mathcal{K}(x_1,x_2|\ell)$ can be generalized to the Coulomb kernel being an operator function acting on the product of fermion fields $\mathcal{K}(x_1,x_2|\ell)\psi(x_1)\bar{\psi}(x_2) \Rrightarrow \mathcal{K}(X,z|\hat{\ell})\psi(z+X/2)\bar{\psi}(-z+X/2)$, where $\hat{\ell}_\mu \sim \partial/\partial X_\mu$ and $X_\mu = \frac{1}{2}(x_1+x_2)_\mu$. The action of this operator can be fully understood in the Fourier analysis.

The next step in our construction is a redefinition of the action (12) in terms of *bilocal fields* $\mathcal{M}(x_1,x_2) = \mathcal{M}(X,z)$ and generalized Coulomb kernel $\mathcal{K}(X,z|\hat{\ell})$ by means of the Legendre transformation [1, 17]

$$\frac{1}{2}\left(\psi_2\bar{\psi}_1,\mathcal{K}_{(1,2)}(\hat{\ell})\psi_1\bar{\psi}_2\right) = -\frac{1}{2}\left(\mathcal{M},\mathcal{K}^{-1}(\hat{\ell})\mathcal{M}\right) + \left(\psi_1\bar{\psi}_2,\mathcal{M}\right). \tag{13}$$

Bilocal fields obey the Markov-Yukawa condition [14]

$$z_\mu\frac{\partial\mathcal{M}(X,z)}{\partial X^\mu} = 0\,. \tag{14}$$

This constraint realized the physical definition of a bound state[2], the so called "simultaneity principle", which states that we can observe experimentally two particles as the bound state $\mathcal{M}(x_1, x_2)$ at one and the same time

$$\mathcal{M}(x_1, x_2) = \mathcal{M}(z, X)\Big|_{\textsf{Atom at rest}} \equiv \mathrm{e}^{\mathrm{i}MX_0}\, \Psi(\mathbf{z})\, \delta(\mathbf{z_0})\,.$$

In addition Eq. (14) has deep mathematical meaning [11, 13] as the constraint of irreducible nonlocal representations of the Poincarè group for an arbitrary bilocal field. Therefore, the relativistic covariant unitary perturbation theory in terms of such relativistic bound states can be constructed [11].

The Markov-Yukawa condition (14) is equivalent to our choice of a reference frame in bilocal dynamics as a vector operator with eigenvalues proportional to the total momenta of bound states [14]. This is a key point of our construction, it means that the constraint-shell QED allows to construct the bound state relativistic covariant perturbation theory. The most straightforward way for constructing QED of instantaneous bound states is to take the free constraint-shell QED action with the generalized Coulomb instantaneous interaction[3](13). Then the instantaneous bound state generating functional $Z^*_{\mathrm{IBS}}[\,\hat{\ell}\,]$ can be defined, which after an integration over fermions has the following form

$$Z^*_{\mathsf{IBS}}[s^*, \bar{s}^*, J^*|\hat{\ell}\,] = \langle *\,|\int \mathcal{D}\mathcal{M}\mathrm{e}^{\mathrm{i}W_{\mathrm{eff}}[\mathcal{M}]+\mathrm{i}S_{\mathrm{eff}}[\mathcal{M}]}|\,*\,\rangle\,, \tag{15}$$

where

$$\begin{aligned} W_{\mathrm{eff}}[\mathcal{M}] &= \left(-\mathrm{i}tr\log(-G_A^{-1}+\mathcal{M})\right) - \frac{1}{2}\Big(\mathcal{M}, \mathcal{K}^{-1}(\hat{\ell})\mathcal{M}\Big)\,, \qquad (16)\\ S_{\mathrm{eff}}[\mathcal{M}] &= \left(s^*\bar{s}^*, (G_A^{-1}-\mathcal{M})^{-1}\right),\\ -G_A^{-1} &\equiv \left(\mathrm{i}\gamma^\mu\partial_\mu - \mathrm{i}e\gamma^\mu A^*_\mu - m\right)\delta^{(4)}(x-y)\,. \end{aligned}$$

The bracket $\langle *\,|\ \dots\ |\,*\,\rangle$ means the averaging over transverse photons

$$\langle *\,|\Upsilon|\,*\,\rangle = \int \prod_{a=1,2} \mathcal{D}A^*_a \mathrm{e}^{\mathrm{i}W[A^*]}\Upsilon.$$

The effective field theory defined by the action $W_{\mathrm{eff}}[\mathcal{M}]$ reproduces the fermion and bound state spectra. Indeed, let us determine the minimum of the effective action (16)

$$\frac{\delta W_{\mathrm{eff}}}{\delta\mathcal{M}}\Big|_{\mathcal{M}=\Sigma} = 0\,,$$

[2] One of the first definitions of the physical bound states in QED belongs to Lord Eddington [4]: "A proton yesterday and an electron today do not make an atom".

[3] This prescription neglects the "retardation" interaction. However, one may argue, that at the point of the existence of the bound state with the definite total momentum any instantaneous interaction (12) with the time axis parallel to this momentum is much greater that any "retardation" interaction [12].

where Σ is the corresponding classical solution for the bilocal field. This stationarity equation can be rewritten in the form of the Schwinger-Dyson equation [15], which describes the energy spectrum of Dirac particles in bound states. Then the effective action can be expanded around the point of minimum[4] $\mathcal{M} = \Sigma + \widetilde{\mathcal{M}}$,

$$W_{\text{eff}}(\Sigma + \widetilde{\mathcal{M}}) = W_{\text{eff}}^{(2)} + \dots . \tag{17}$$

The small fluctuations $\widetilde{\mathcal{M}}$ can be represented as a sum over the complete set of orthonormalized solutions Γ of the equation

$$\frac{\delta^2 W_{\text{eff}}}{\delta \mathcal{M}^2}\Big|_{\mathcal{M}=\Sigma} \Gamma = 0 \, .$$

This equation reproduces the Bethe-Salpeter equation [19], which describes the spectrum of bound states (see Ref. [15]).

4. *Summary and open problems*

The relativistic invariant generating functional for instantaneous bound states (15) and their amplitudes constructed in this paper by means of the operator generalization of the initial data in Dirac's radiation variables states the following problems: 1) building of a bound state S-matrix; 2) a proof of renormalizability. The proof of renormalizability can be achieved, because the main difference of IBS functional from the FP functional for Lorentz gauge formulation is only the source term. We hope that obtained IBS functional can be useful for studying the physics of the QED bound states like positronium, and it can help us to clear up the QCD hadronization problems.

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References

[1] J. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **10** (1974) 2428; H. Kleinert, Phys. Lett. B **26** (1976) 429; E. Schreiner, Phys. Rev. D **16** (1977) 1877.

[2] P. A. M. Dirac, Proc. Roy. Soc. A **114** (1927) 243; Can. J. Phys. **33** (1955) 650.

[3] P. A. M. Dirac: *Lectures on Quantum Mechanics*, New York (1964).

[4] A. Eddington: *Relativistic Theory of Protons and Electrons*, Cambridge University Press (1936).

[4] The higher terms in the expansion (17) describe the interactions of bilocal fields. If the Coulomb potential is replaced by any rising potential (QCD), these terms are responsible for the spontaneous breakdown of chiral symmetry [16].

[5] L. Faddeev, Teor. Mat. Fiz. **1** (1969) 3.

[6] L. Faddeev and V. Popov, Phys. Lett. B **25** (1967) 29.

[7] E. Fermi, Rev. Mod. Phys. **4** (1932) 87.

[8] S. A. Gogilidze, V. N. Pervusin and A. M. Khvedelidze, Phys. Part. Nucl. **30** (1999) 66.

[9] W. Heisenberg and W. Pauli, Z. Phys. **56** (1929) 1; Z. Phys. **59** (1930) 166.

[10] W. Kummer, Nuovo Cim. **31** (1964) 219.

[11] L. Y. Kalinovsky et al., Few Body Systems **10** (1991) 87.

[12] S. Love, Ann. Phys. **113** (1978) 153.

[13] J. Lukierski and M. Oziewicz, Phys. Lett. B **69** (1977) 339.

[14] M. A. Markov, J. Phys. (USSR) **3** (1940) 453; H. Yukawa, Phys. Rev. **77** (1950) 219.

[15] V. N. Pervushin, Phys. Part. Nucl. **34** (2003) 348.

[16] V. N. Pervushin, Nucl. Phys. B **15** (Proc. Supp.) (1990) 197.

[17] V. N. Pervushin, H. Reinhardt and D. Ebert, Sov. J. Part. Nucl. **10** (1979) 1114.

[18] I. V. Polubarinov, Phys. Part. Nucl. **34** (2003) 741.

[19] E. E. Salpeter, Phys. Rev. D **87** (1952) 328.

[20] B. Zumino, J. Math. Phys. **1** (1960) 1.

FUNKCIONAL ZA TVORBU VEZANIH STANJA U QED

Očito Lorentz-kovariantna formulacija kvantne elektrodinamike zanemaruje trenutno Coulombovo međudjelovanje i njegovu posljedicu – trenutna vezana stanja. U ovom se radu razmatra postavljanje tvorbenog funkcionala za trenutno vezana stanja primjenom operatora poopćenih početnih stanja u Dirac-Hamiltonovom pristupu QED.