Printed ISSN 1330–0016
Online ISSN 1333–9133
CD ISSN 1333–8390
CODEN FIZBE7

PERIASTRON SHIFT IN THE ORBIT OF A PARTICLE ORBITING A KERR BLACK HOLE

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Received 2 October 2005; Accepted 25 September 2006
Online 29 December 2006

In this paper, we present a simple calculation of periastron shift in the orbit of a test particle in the Kerr field. We differentiate the prograde and retrograde orbits by the orbital angular momentum of the test particle. The orbit equations are then clearly different for the two types of motion. The calculated periastron shift is larger in a retrograde orbit than in a prograde orbit, a fact confirming previous results. We compare the difference between periastron shifts in the two types of motion with the results of Esteban and Diaz who used elliptic integrals to calculate the shift.

PACS numbers: 04.20.Cv, 04.25.-g UDC 521.184
Keywords: periastron shift, Kerr field, prograde and retrograde orbits

In an astrophysical binary system, the orbit of a star around a massive central body is not likely be a simple Keplerian orbit due to general relativistic effects. In particular, a periastron shift will result in rosetta shaped orbits. Since the magnitude of periastron advance strongly depends on the compactness of the central body, the detection of such an effect is likely to provide information about the nature of the central body. This could be the case of stars orbiting close to the center of our Galaxy, where a massive black hole is presumed to reside. This massive body could be a static Schwarzschild black hole or a stationary Kerr black hole. Other possibilities are there, too. In any case, the study of periastron shifts in orbits in various types of gravitational fields has become important.

In a binary system, the periastron shift is due to Newtonian and relativistic contributions. The Newtonian effects on the periastron shift are well known. A detailed discussion of this can be found in Ref. [1]. Regarding the relativistic effects on periastron shift, we know that the dominant contribution was found by

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Einstein [2] in the calculation of Mercury's perihelion shift. Compact calculation of Mercury's perihelion shift can be found in Ref. [3]. Damour and Schafer [4] found an exact solution useful to calculate, in the test-particle approximation, the relativistic periastron shift in binaries in which the central body could be a static black hole. Esteban and Diaz [5] obtained analytical solutions for the relativistic periastron shift including the spin-orbit interaction. They, therefore, have considered a binary system composed of a rotating black hole and an orbiting star. They have used the Kerr metric to model the exterior space-time of the rotating black hole and assumed test-particle approximation. In two previous articles [6 – 7], we have calculated the periastron shift in Kerr space-time in the orbit of an effective particle. Recently, Bini et al. [8] calculated periastron shift in Kerr space-time in the test-particle approximation.

The purpose of this paper is to clarify the effect of spin-orbit interaction on periastron shift in prograde and retrograde orbits. The analysis presented by Esteban and Diaz [5] shows that the spin-orbit interaction diminishes (increases) the value of periastron shift when the star in the binary system orbits in the same (opposite) direction as the rotating black hole. The calculation of Bini et al. [8] does not refer to the senses of the rotations of the bodies in the binary system. Our calculations in Ref. [6 – 7] agree with the conclusion drawn in Ref. [5], that is, the periastron shift in prograde orbit is smaller than that in retrograde orbit. In literature, there are several ways for taking into account the relative senses of rotations of the bodies in a binary system. Let the spin of the central body be denoted by S and the orbital angular momentum of the orbiter be J. In Ref. [5], the retrograde orbit is marked by a negative S and the prograde orbit by a positive S. In our calculations in Ref. [6 – 7], we have marked the prograde orbit by a positive value of S/J and the retrograde orbit by a negative value of S/J. Signs of the product SJ are also in use for this purpose [2]. One may thus be confused with the results. In the present paper, we shall present a straightforward calculation to support the conclusion of our previous result [6 – 7] and that of Esteban and Diaz [5].

Let us consider a rotating black hole of mass M around which a star (considered as a test particle) of mass m is moving. Let spherical polar coordinates, r, θ, φ, centered on the black hole, be used to describe the location of the particle, with the hole's spin, S, along the polar axis. The Lagrangian for the motion of the particle is given [9 – 10], to linear order in S but otherwise in solely Newtonian theory, by

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\,\dot{\varphi}^2\right) + \frac{GmM}{r} - \frac{2GmS\sin^2\theta}{c^2 r}\dot{\varphi}\,, \tag{1}$$

where an over dot represents differentiation with respect to time. To leading order in S and in M/r, the motion resulting from this Lagrangian is the same as in the Kerr metric. Note, however, that this Lagrangian ignores M/r corrections that are present in the Schwarzschild space-time. In particular, if we set $S = 0$ in Eq. (1), the remaining Lagrangian is Newtonian and hence, will not give rise to the periastron shift that results in Schwarzschild field. We shall come to this point again shortly. Now, we restrict our consideration to motion in the equatorial plane.

The Lagrangian (1) then reduces to

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GmM}{r} - \frac{2GmS}{c^2 r}\,\dot{\varphi}\,. \tag{2}$$

There are two constants of motion [9]

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{GmM}{r}\,, \tag{3}$$

$$J = mr^2\dot{\varphi} - \frac{2GmS}{c^2 r}\,, \tag{4}$$

where E is the energy of the test particle, and J is the z-component (component along the spin axis of the hole) of orbital momentum. We now differentiate the two cases of orbital angular momentum: J as given by Eq. (4) corresponds to the prograde orbit, and J as given by the negative of the expression in Eq. (4) corresponds to the retrograde orbit. Hence, we have

$$J = mr^2\dot{\varphi} - \frac{2GmS}{c^2 r}\,, \quad \text{prograde orbit,} \tag{5}$$

$$J = \frac{2GmS}{c^2 r} - mr^2\dot{\varphi}\,, \quad \text{retrograde orbit,} \tag{6}$$

where both of J's are positive in magnitude. Now, the Euler-Lagrange equation associated with r reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0\,, \tag{7}$$

and we obtain, substituting L from Eq. (2) into Eq. (7),

$$m\left(\ddot{r} - r\dot{\varphi}^2\right) = -\frac{GmM}{r^2} + \frac{2GmS}{c^2 r^2}\,\dot{\varphi}\,, \tag{8}$$

which is valid for both retrograde and prograde motion. Next, we solve Eq. (5) and Eq. (6) for $\dot{\varphi}$ and substitute those in Eq. (8) to obtain differential equations for retrograde and prograde motions

$$\ddot{r} - \frac{J^2}{m^2 r^3} = -\frac{GM}{r^2} + \frac{6GJS}{mc^2 r^4}\,, \quad \text{prograde orbit,} \tag{9}$$

$$\ddot{r} - \frac{J^2}{m^2 r^3} = -\frac{GM}{r^2} - \frac{6GJS}{mc^2 r^4}\,, \quad \text{retrograde orbit,} \tag{10}$$

where we have neglected terms proportional to S^2. Following the standard procedure, we now use the variable $u = 1/r$, and take φ as the independent variable. We obtain for both retrograde and prograde motion

$$\ddot{r} \cong -\frac{J^2}{m^2} u^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2}, \tag{11}$$

where we have used the criterion, $\frac{2GSm}{c^2 Jr} \ll 1$, which we assume valid for orbits far from the central body. Using (11) in Eqs. (9) and (10), we get the familiar type of orbit equations. However, the resulting equations do not contain the familiar term $\frac{3GM}{c^2}u^2$ responsible for periastron shift in Schwarzschild field. Note that in the Schwarzschild field, we have the orbit equation [11]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{Gm^2 M}{J^2} + \frac{3GM}{c^2} u^2. \tag{12}$$

However, Eqs.(9) and (10) do not give rise to the 2$^{\mathrm{nd}}$ term on the right hand side of Eq. (12). This happens due to the Newtonian form of the Lagrangian (2) once $S = 0$ is used. Since our intention is to obtain the correct periastron shift, we put this term in the orbit equation. Therefore, we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{Gm^2 M}{J^2} + \frac{3GM}{c^2} u^2 - \frac{6GmS}{Jc^2} u^2, \quad \text{prograde orbit,} \tag{13}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u = \frac{Gm^2 M}{J^2} + \frac{3GM}{c^2} u^2 + \frac{6GmS}{Jc^2} u^2, \quad \text{retrograde orbit,} \tag{14}$$

Using the specific spin of the hole given by $a = S/Mc$, and putting $J^2/(Gm^2 M) = p$ (the semi-letus rectum of the orbit), we obtain

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u - \frac{1}{p} = \frac{3GM}{c^2}\left(1 - \frac{2amc}{J}\right) u^2, \quad \text{prograde orbit,} \tag{15}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\varphi^2} + u - \frac{1}{p} = \frac{3GM}{c^2}\left(1 + \frac{2amc}{J}\right) u^2, \quad \text{retrograde orbit.} \tag{16}$$

Solution to Eqs. (15) and (16) can be found using the standard procedure [12], and the subsequent periastron shift can be found. We obtain

$$\delta\varphi \cong \frac{6\pi GM}{c^2 p}\left(1 - \frac{2amc}{J}\right), \quad \text{prograde orbit,} \tag{17}$$

$$\delta\varphi \cong \frac{6\pi GM}{c^2 p}\left(1 + \frac{2amc}{J}\right), \quad \text{retrograde orbit,} \tag{18}$$

Now, $J = m\sqrt{GMd(1-e^2)}$ and $p = d(1-e^2)$. Substituting these values into Eqs. (17) and (18), we finally obtain

$$\delta\varphi \cong \frac{6\pi GM}{c^2 d(1-e^2)} - \frac{12\pi a\sqrt{GM}}{cd^{3/2}(1-e^2)^{3/2}}\,, \quad \text{prograde orbit,} \tag{19}$$

$$\delta\varphi \cong \frac{6\pi GM}{c^2 d(1-e^2)} + \frac{12\pi a\sqrt{GM}}{cd^{3/2}(1-e^2)^{3/2}}\,, \quad \text{retrograde orbit,} \tag{20}$$

The first term on the right hand sides of Eqs. (19) and (20) is the familiar periastron shift originating from Schwarzschild gravitoelectric field. The second term is the lowest-order contribution from spin-orbit interaction (gravitomagnetic contribution). The total shift is larger in the retrograde orbit than that in the prograde orbit. The contribution from spin-orbit interaction is equal in magnitude but opposite in sign in the two types of motion. Moreover, this gravitomagnetic contribution is very close in magnitude to that in our previous calculations for effective particle [6−7]. Moreover, we get results close to those obtained by Bini et al. [8]; the difference lies in the factor 12, which in Ref. [8] is 16. To compare our results with those of Esteban and Diaz [5], we calculate the differences of periastron shifts between the two types of motion using semi-major axis and eccentricity values of Mercury's orbit around the Sun and use mass values from 1 solar mass to 10^6 solar masses. The results of our calculation are shown in Table 1 and those of Esteban and Diaz [5] are shown in Table 2.

Table 1. For the given values of a/M, this table shows the differences between the periastron shifts in retrograde and prograde orbits according to Eqs. (19) and (20). The orbital parameters are assumed to be the same as those of Mercury (orbit around Sun, semi-major axis =0.3871 AU, eccentricity = 0.2056). $M_\odot$ is the solar mass. Numbers in a column should be multiplied by the power of 10 above it.

$\frac{a}{M}$	Mass of the central object						
	$1\ M_\odot$	$10\ M_\odot$	$10^2\ M_\odot$	$10^3\ M_\odot$	$10^4\ M_\odot$	$10^5\ M_\odot$	$10^6\ M_\odot$
	$(\times 10^{-11})$	$(\times 10^{-9})$	$(\times 10^{-8})$	$(\times 10^{-6})$	$(\times 10^{-5})$	$(\times 10^{-3})$	$(\times 10^{-2})$
1	32.726	10.349	32.726	10.349	32.726	10.349	32.726
4/5	26.181	8.2793	26.181	8.2793	26.181	8.2793	26.181
3/5	19.636	6.2095	19.636	6.2095	19.636	6.2095	19.636
2/5	13.090	4.1396	13.090	4.1396	13.090	4.1396	13.090
1/5	6.5453	2.0698	6.5453	2.0698	6.5453	2.0698	6.5453

Table 2. Results of Esteban and Diaz [5].

$\frac{a}{M}$	Mass of the central object						
	$1\ M_\odot$	$10\ M_\odot$	$10^2\ M_\odot$	$10^3\ M_\odot$	$10^4\ M_\odot$	$10^5\ M_\odot$	$10^6\ M_\odot$
	$(\times 10^{-11})$	$(\times 10^{-9})$	$(\times 10^{-8})$	$(\times 10^{-6})$	$(\times 10^{-5})$	$(\times 10^{-3})$	$(\times 10^{-2})$
1	21.793	6.89	21.75	6.895	21.852	7.0624	28.4397
4/5	17.44	5.51	17.37	5.516	17.482	5.6499	22.7122
3/5	13.08	4.13	12.99	4.137	13.112	4.2374	17.0111
2/5	8.72	2.75	8.72	2.758	8.741	2.8249	11.3298
1/5	4.36	1.45	4.36	1.379	4.37	1.4125	5.6617

We see that our values are larger by about 50% than those of Ref. [5] for smaller masses, but our values are very close to those of Ref. [5] for 10^6 solar masses. If the formula of Bini et al. [8] is used, the values will be more than those of ours by about 25% (i.e., 75% larger than those of Ref. [5]). So, we conclude that the approximate formulae (19) and (20) give periastron shifts for equatorial prograde and retrograde orbits that are appropriate for compact central bodies of masses $\geq 10^6$ solar masses.

We now summarize: We have employed Euler-Lagrange equation, using the Lagrangian given in Refs. [9 − 10] that is appropriate for motion in Kerr space-time, to calculate the lowest-order periastron shifts in prograde and retrograde equatorial orbits of a test particle orbiting a Kerr black hole. We have cast the orbital angular momentum of the orbiting particle in two distinct forms, one algebraically opposite to the other, to differentiate orbit equations for prograde and retrograde orbits. The periastron shift in retrograde orbit is found to be increased by the spin-orbit interaction over the gravitoelectric contribution. In prograde orbit, the shift is lowered by the spin-orbit interaction. These facts are in agreement with previous results of Esteban and Diaz [5] and of ours [6 − 7]. The work in this article clarifies the matter of sign of the contribution of spin-orbit interaction in the periastron shift in equatorial orbits of a particle orbiting a Kerr black hole. The formulae we obtain, Eqs. (19) and (20), give periastron shifts for prograde and retrograde orbits of a test body orbiting a spinning central compact star or a blak hole of mass $\geq 10^6$ solar masses.

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POMAK PERIASTRONA STAZE ČESTICE KOJA KRUŽI OKO KERROVE CRNE RUPE

Predstavljamo jednostavan račun pomaka periastrona staze čestice u Kerrovom polju. Razlikujemo istosmjerno i protusmjerno kruženje prema impulsnom momentu čestice. Na taj se način jednadžbe staze jasno razlikuju za te načine kruženja. Izračunat pomak periastrona veći je za protusmjerno nego za istosmjerno kruženje, što potvrđuje ranije račune. Uspoređujemo razlike pomaka periastrona za ta dva načina kruženja s rezultatima Estebana i Diaza koji su primijenili eliptičke integrale za računanje pomaka.