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COLOURED S-WAVE QUARK CLUSTERS WITH FLAVOUR SYMMETRY BREAKING

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Dedicated to the memory of Professor Dubravko Tadić

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We study the properties of coloured three-particle s-wave quark clusters when flavour symmetry is broken. The relevance of such clusters for models of pentaquarks is shortly mentioned.

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1. *Introduction*

For the last thirty years, colour magnetism has been a major tool in understanding hadron spectroscopy. It is still believed by many that the exchange of coloured gluons provide the easiest way of explaining the mass differences between hadronic states made up of the same (valence) quarks [1]. When all spatial degrees of freedom are integrated out, we have an interaction Hamiltonian over colour spin space which is the usual

$$H_{\rm CM} = -\sum_{i,j} C_{ij} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j \boldsymbol{\sigma}_i . \boldsymbol{\sigma}_j \, . \tag{1}$$

Here the coefficients C_{ij} are, among other things, dependent on the quark masses and properties of the spatial wave functions of the quarks and the antiquark in the system. The solution of the eigenvalue problem of the Hamiltonian above is therefore of interest, not only in spectroscopy, but in all reactions where an antiquark or a quark interact with a system of other quarks.

For this reason we think it is appropriate to present a full analysis of the colour-magnetic interaction with complete flavour symmetry breaking for the (qqq) and $(qq\overline{q})$ systems. It is a well defined problem of undeniable interest, and we shall see that flavour symmetry breaking gives effects that are far more subtle than in the (trivial) diquark $(q\overline{q})$ system.

The diagonalizing of $H_{\rm CM}$ in the cases where baryons are made of three valence quarks and mesons of $(q\overline{q})$ pairs, all coupled to a colour singlet, has been a standard subject of textbooks for a long time. For hadrons with more complicated structure, where one adds extra $(q\overline{q})$ pair(s) to the simplest structures, new states ("multiquark states") are created, some of which carrying quantum numbers that cannot be explained with baryons made of three valence quarks and mesons made of $(q\overline{q})$ pairs. These are exotic states.

After the recent reports of possible observations of exotic baryons [2], the interest in multiquark states has increased very much [3 − 13]. An extensive list of references can be found in Ref. [12].

The search for such states had been strongly encouraged by Diakonov, using predictions from the chiral soliton model [13].

Studies of baryons with more than three quarks go back more than a quarter of a century. At that time, one made models where "coloured ions" were bound together by colour-electric flux tubes [14] . The mass defects due to the colour-magnetism were mostly made in the flavour symmetric limit. A group-theoretical mass formula was applied, the mass defect being then expressed in terms of the quadratic Casimir operators for the $SU(2)$-spin, the $SU(3)$-colour and the $SU(6)$-colourspin group [15, 16]. In the cases where colour-spin, colour and spin for quarks and antiquarks can be simultaneously quantized together with the same operators for the whole system, the results are quite easily generalized to flavour symmetry breaking. In other cases not.

As can be immediately noticed by elementary group theory computations, some multiquark states are *mixtures* of states of colour $SU(3)_c$, spin $SU(2)_s$ and colour spin $SU(6)_{cs}$ representations, and that implies some care in their treatment. Such a situation appears as soon as a set of quarks and antiquarks is considered, and these are among the cases of s-wave clusters that we address in this paper.

From the decomposition of $SU(3)$ representations

$$3 \times 3 \times 3 = 1 + 8 + 8 + 10 \tag{2}$$

and

$$3 \times 3 \times \overline{3} = 3 + \overline{6} + 3 + 15\,, \tag{3}$$

one deduces that the possible s-wave triplets of quarks belong to one of the following colour representations $c = 1, 8, 10$, while clusters made of $2q$ and $1\overline{q}$ stand in $c = 3, 6$ or 15, the total spin being for each cluster $s = 1/2$ and $s = 3/2$, respectively.

Let us add that quark triplets in colour octet $(qqq)^8_s$ as well as $(qq\overline{q})^{\overline{6}}_s$ configurations have been considered a long time ago in some detail, mostly in the flavour symmetry limit, as part of exotic baryons. Such "pentaquarks", earlier called "meso-baryonium" [16], were constituted by two clusters of the type: $(qqq)^8 - (q\overline{q})^8$ or $(qq\overline{q})^{\overline{6}} - (qq)^6$, each cluster standing in an s-wave, and separated from the other by an orbital angular momentum barrier. The configuration $(qq\overline{q})^3_{1/2} - (qq)^{\overline{3}}_0$, again with a relative $L = 1$, has also been recently used [4] to make pentaquark states of spin $1/2^+$: a detailed analysis of this approach, based from a study of $H_{\rm CM}$ with flavour symmetry breaking, has been made in Ref. [11].

We present our results over a set of base states. We first couple two quarks q_1 and q_2 into a definite colour and spin system, then add a third quark/antiquark and consider the resulting clusters with definite colour and spin. Note that different orderings of quarks in a cluster with specified colour c and spin s generally lead to different Hamiltonian matrices. For example, choosing u and d as quarks q_1 and q_2 in the (udc) system will provide a Hamiltonian matrix different from the one obtained by choosing the quarks d and c for the same positions 1 and 2. The eigenvalues will of course be the same, as well as the physical content of the eigenvectors.

However, is not quite without importance how one numbers the quarks: the reduction of the Hamiltonian obtained by imposing some (approximate) flavour symmetry, such as isospin symmetry, is explicit if u and d are chosen as quarks 1 and 2. In the isospin symmetry case, we have noted that our calculations are in agreement, as could be expected, with earlier results in the three flavour sector [17]. For quarks with identical flavour, the effect of the Pauli exclusion principle is most easily incorporated by considering them as q_1 and q_2.

When we sometimes refer to $SU(3)$ flavour representations, it evidently does not mean that we consider u, d and s quarks only, the number 3 comes just because we study three-particle systems, so that the (maximal) number of different flavours in a cluster is 3.

The eigenvectors of $H_{\rm CM}$ are usually not falling into one specific representation of $SU(3)_f$, they are naturally (also called magically) mixed combinations of different $SU(3)_f$ representations.

2. *The* $(qqq)^8_s$ *triquark*

This type of triquark had shown up in one of the two favorite configurations - the second one implying the $(qq\overline{q})^{\overline{6}}_s$ cluster - proposed in Ref. [16] for narrow exotic baryons made of two clusters protected one from the other by a relative angular momentum barrier.

Let us denote by q_1, q_2 and q_3 the three quarks under study, all in a relative s-wave.

i) We consider first the case of total spin $s = 1/2$, that is $|q_1q_2q_3\rangle^8_{1/2}$. Reminding of the following decompositions of $SU(3)$ representations:

$$3 \times 3 = \overline{3} + 6 \; ; \qquad \overline{3} \times 3 = 1 + 8 \; ; \qquad 6 \times 3 = 8 + 10,$$

on which the Hamiltonian $H_{\rm CM}$ acts is four dimensional, and a natural basis is provided with the four states:

$$\begin{aligned} \chi_1 &= |(q_1q_2)^6_1\rangle \otimes |(q_3)^3_{1/2}\rangle \,, \\ \chi_2 &= |(q_1q_2)^{\overline{3}}_1\rangle \otimes |(q_3)^3_{1/2}\rangle \,, \\ \chi_3 &= |(q_1q_2)^6_0\rangle \otimes |(q_3)^3_{1/2}\rangle \,, \\ \chi_4 &= |(q_1q_2)^{\overline{3}}_0\rangle \otimes |(q_3)^3_{1/2}\rangle \,, \end{aligned} \tag{4}$$

the notation here being

$$\chi_1 = |(q_1q_2)^c_s\rangle \otimes |(q_3)^3_{1/2}\rangle \,, \tag{5}$$

where c is the colour and s the spin of the doublet of the two quarks q_1 and q_2.

The states χ_1 and χ_4 have the pair of two quarks (q_1q_2) which are coupled symmetrically in colour-spin and are therefore belonging to the $(6 \times 6)_S = 21$-dimensional representation of $SU(6)_{cs}$, the states χ_2 and χ_3 are antisymmetric in colour-spin of the two quarks (q_1q_2) and fall in the $(6 \times 6)_A = 15$-dimensional representation. Note that if the two quarks q_1 and q_2 are identical in flavour, the states χ_1 and χ_4 vanish due to the Pauli principle.

A way to explicitly compute the 4×4 matrix representing $H_{\rm CM}$ relative to the $(qqq)^8_{1/2}$ triplet is to study separately the colour part and the spin part namely:

$$H_{\rm C} = -\sum_{i,j} C_{ij} \overrightarrow{\lambda}_i \cdot \overrightarrow{\lambda}_j \; , \qquad H_{\rm S} = -\sum_{i,j} C_{ij} \overrightarrow{\sigma}_i \cdot \overrightarrow{\sigma}_j \; , \tag{6}$$

and then to perform a kind of "tensor product" of the two so-obtained 2×2 matrices.

Let us consider the colour-action part. Then, when acting by $H_{\rm C}$ on the χ_i's, it will be convenient to express $|(q_1q_2)^c(q_3)^3\rangle^8$ where $c = 6$ or $\overline{3}$ in terms of $|(q_1q_3)^c(q_2)^3\rangle^8$ and $|(q_2q_3)^c(q_1)^3\rangle^8$ (we omit the lower spin index in this computation). By direct calculation, one obtains the colour crossing

$$\begin{aligned} V_c \equiv \begin{pmatrix} |(q_1q_2)^6(q_3)^3\rangle^8 \\ |(q_1q_2)^{\overline{3}}(q_3)^3\rangle^8 \end{pmatrix} &= \begin{bmatrix} -\frac{1}{2} & \sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} & -\frac{1}{2} \end{bmatrix} \begin{pmatrix} |(q_2q_3)^6(q_1)^3\rangle^8 \\ |(q_2q_3)^{\overline{3}}(q_1)^3\rangle^8 \end{pmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \sqrt{\frac{3}{2}} \\ \sqrt{\frac{3}{2}} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} |(q_1q_3)^6(q_2)^3\rangle^8 \\ |(q_1q_3)^{\overline{3}}(q_2)^3\rangle^8 \end{pmatrix}, \end{aligned} \tag{7}$$

from where we can also derive the (inverse) expressions of $|(q_1q_3)^c(q_2)^3\rangle^8$ and $|(q_2q_3)^c(q_1)^3\rangle^8$ in terms of $|(q_1q_2)^c(q_3)^3\rangle^8$. It is then straightforward to derive the $H_{\rm C}$ matrix.

A similar technique will allow to construct the 2×2 $H_{\rm S}$ matrix, and we finally give the complete expression for the colour magnetic Hamiltonian $H_{\rm CM}$ acting on the 4-dimensional vector $\overrightarrow{\chi} = (\chi_1, \chi_2, \chi_3, \chi_4)$

$$H_{\rm CM} =$$

$$- \begin{bmatrix} \frac{4}{3}\,C_{12}+\frac{10}{3}(C_{13}+C_{23}) & 2\sqrt{3}\,(C_{13}-C_{23}) & \frac{5}{\sqrt{3}}\,(C_{13}-C_{23}) & 3\,(C_{13}+C_{23}) \\ 2\sqrt{3}\,(C_{13}-C_{23}) & -\frac{8}{3}\,C_{12}-\frac{2}{3}\,(C_{13}+C_{23}) & 3\,(C_{13}+C_{23}) & -\frac{1}{\sqrt{3}}\,(C_{13}-C_{23}) \\ \frac{5}{\sqrt{3}}\,(C_{13}-C_{23}) & 3\,(C_{13}+C_{23}) & -4\,C_{12} & 0 \\ 3\,(C_{13}+C_{23}) & -\frac{1}{\sqrt{3}}\,(C_{13}-C_{23}) & 0 & 8\,C_{12} \end{bmatrix}. \tag{8}$$

In the flavour-symmetry limit, $C_{12} = C_{23} = C_{31} = C$, and $H_{\rm CM}$ reduces to

$$H_{\rm CM} = -C \cdot \begin{bmatrix} 8 & 0 & 0 & 6 \\ 0 & -4 & 6 & 0 \\ 0 & 6 & -4 & 0 \\ 6 & 0 & 0 & 8 \end{bmatrix}, \tag{9}$$

and the space state decomposes into two invariant subspaces. One is spanned by χ_1 and χ_4, its $SU(3)$ flavour content is a singlet and an octet with eigenvalues $-14C$ and $-2C$ for $H_{\rm CM}$. The other, spanned by χ_2 and χ_3, contains a flavour octet and a decuplet where the eigenvalues of $H_{\rm CM}$ are $-2C$ and $+10C$ respectively.

Let us note that the determination of the associated $SU(3)$ flavour representations is naturally obtained from the corresponding colour-spin $SU(6)_{cs}$ ones. Indeed, in $SU(6)$

$$6 \times 6 = 21 + 15\,, \tag{10}$$

with the $SU(3)_c \times SU(2)_s$ decompositions:

$$6 = (3, 1/2)\ ; \qquad 21 = (6,1) + (\overline{3}, 0)\ ; \qquad 15 = (6,0) + (\overline{3}, 3)$$

(in which the $SU(3)$ part is denoted by its dimension and the $SU(2)$ part by its s-label), and we remark that the couples (q_1q_2) in χ_1 and χ_4 belong to the symmetric 21 representation of $SU(6)_{cs}$, while in χ_2 and χ_3 they stand in the antisymmetric 15 one. Moreover,

$$21 \times 6 = 56 + 70 \quad \text{and} \quad 15 \times 6 = 20 + 70\,. \tag{11}$$

Then the corresponding $SU(3)$ multiplets are selected by insuring the complete antisymmetry of $SU(6)_{cs} \times SU(3)_f$. As noted before, if two quarks are identical in flavour, it is convenient to label them as particles 1 and 2 so that the states χ_1 and χ_4 are forbidden by the Pauli principle.

In this case, and in all other cases where we have a system of only quarks (or only antiquarks), the eigenstates in the flavour symmetric limit (only!) correspond to sharp values of the total colour-spin. Examining the $SU(3)_c \times SU(2)_s$ decompositions:

$$56 = (8, 1/2) + (10, 3/2)\,,$$

$$70 = (8, 1/2 + 3/2) + (10 + 1, 1/2)\,,$$

$$20 = (8, 1/2) + (1, 3/2)\,,$$

we note that the eigenvectors, which are mixtures of our base states, are identical to the 56 and 70 representations as well as of the 70 and 20 representations of $SU(6)_{cs}$, any of these $SU(6)$ representations containing an $SU(3)$ colour octet and an $SU(2)$ spin doublet. When the flavour symmetry is broken this is no longer the case, and $H_{\rm CM}$ and the total colour-spin can no longer be simultaneously quantized.

Finally, it is interesting to remark that, if the three particles are identical, then only the combination (χ_2 - χ_3) is allowed by the Pauli principle, as can be seen hereafter. Indeed, the flavour sector for the three quarks must then be the symmetric 10 dimensional representation of $SU(3)_f$, implying corresponding states to belong to the completely antisymmetric 20 dimensional representation of $SU(6)_{cs}$. As explicited just above, such a configuration involves for the doublet $(q_1 q_2)$ the 15 of $SU(6)_{cs}$ and the product 15×6 (see Eq. (11)). But a direct Clebsch-Gordan computation can show that the combination (χ_2 - χ_3) belongs exactly to the 20 of $SU(6)_{cs}$, while $(\chi_2 + \chi_3)$ belongs to the 70 one. An indirect check of this result can be obtained from the $H_{\rm CM}$ matrix, which immediately reduces to a 2×2 matrix, χ_1 and χ_4 being forbidden as already remarked

$$H_{\rm CM} = -\begin{bmatrix} -4\,C & 6\,C \\ 6\,C & -4\,C \end{bmatrix}. \tag{12}$$

Diagonalizing $H_{\rm CM}$ is immediate and provides the two eigenvalues $10C$ and $-2C$ corresponding to the eigenvectors (χ_2 - χ_3) and $(\chi_2 + \chi_3)$, respectively. Now, referring to our old computation of Ref. [16] where, in the symmetry-limit case, expectation values of $H_{\rm CM}$ have been computed for the three-quark cluster Θ_f(c,s), we recognize the H eigenvalue $10C$ for $\Theta_{10}(8,2)$ with Θ_{10} transforming under the 20 of $SU(6)_{cs}$ and $-2C$ for $\Theta_8(8,2)$ with Θ_8 transforming under the 70 of $SU(6)_{cs}$ (appearing twice).

ii) The case $|q_1 q_2 q_3\rangle^8_{3/2}$ is simpler to study since there is no "spin mixing", and only the colour part of $H_{\rm CM}$ is not trivial, that is

$$H_{\rm CM} = -\begin{bmatrix} \frac{4}{3}\,C_{12} - \frac{5}{3}(C_{13} + C_{23}) & -\sqrt{3}\,(C_{13} - C_{23}) \\ -\sqrt{3}\,(C_{13} - C_{23}) & -\frac{8}{3}\,C_{12} + \frac{1}{3}\,(C_{13} + C_{23}) \end{bmatrix}, \tag{13}$$

acting on the two-dimensional space spanned by χ_1 and χ_2.

3. *The $(qqq)^c_s$ clusters with colour $c = 10$ and $c = 1$.*

These are examples in which the calculation of the colour-magnetic Hamiltonian is very simple. For example, in order to form a colour decuplet, respectively a colour singlet, of three quarks, any pair of quarks must be in a sextet, respectively antitriplet, representation of $SU(3)_c$. A direct computation gives for $\sum_{i,j} \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$ the value

$$K_6 = 4/3 \tag{14}$$

for the sextet and

$$K_{\overline{3}} = -8/3 \tag{15}$$

for the (anti)triplet.

When the total spin of the cluster is $s = 1/2$, then over the basis

$$\pi_1 = |(q_1 q_2)^{c'}_1\rangle \otimes |(q_3)^3_{1/2}\rangle^c \tag{16}$$

$$\pi_2 = |(q_1 q_2)^{c'}_0\rangle \otimes |(q_3)^3_{1/2}\rangle^c \,,$$

with $c = 10$ and $c' = 6$ or $c = 1$ and $c' = \overline{3}$, using the spin crossing matrix, one finds

$$H_{\rm CM} = -K_c \left[\begin{array}{cc} C_{12} - 2(C_{13} + C_{23}) & -\sqrt{3}\,(C_{13} - C_{23}) \\ -\sqrt{3}\,(C_{13} - C_{23}) & -3C_{12} \end{array} \right], \tag{17}$$

with $K_6 = 4/3$ for the $(qqq)^{10}_{1/2}$ cluster and $K_{\overline{3}} = -8/3$ for the usual baryons $(qqq)^1_{1/2}$.

The Pauli principle forbids the state π_2 (π_1) if two quarks chosen as q_1 and q_2 have identical flavour and the cluster is a colour singlet (colour decuplet). If the flavor is identical for all three quarks, then both π_2 and π_1 are forbidden.

If the total spin is $s = 3/2$, then there is only one state, i.e. π^c_1, and $H_{\rm CM}$ reduces to :

$$H_{\rm CM} = -K_c(C_{12} + C_{23} + C_{13})\,. \tag{18}$$

Three-quark clusters in colour decuplet belong to $SU(3)$ flavour octets when $s = 1/2$ and $SU(3)$ flavour singlets when $s = 3/2$. Similarly, three-quark clusters in colour singlet are naturally connected to flavour octets when $s = 1/2$ and to flavour decuplets when $s = 3/2$.

In the flavour symmetry limit, one gets as $H_{\rm CM}$ eigenvalues $+4C$ (respectively $-4C$) for $(qqq)^8_{1/2}$ (respectively $(qqq)^8_{3/2}$) clusters, and $-8C$ (respectively $+8C$) for $(qqq)^1_{1/2}$ (respectively $(qqq)^1_{3/2}$) clusters.

4. The $(qq\bar{q})_s^3$ triquark

This type of triquark state, in the case $s = 1/2$, has recently been used [4] , together with a spin zero diquark state carrying colour $\bar{3}$, to make pentaquark states of spin $1/2^+$ when the triquark and diquark are separated by a $L = 1$ orbital angular momentum. A detailed study of the flavour symmetry breaking has been already performed in our paper [11]. Therefore, we will rapidly provide hereafter with the colour-magnetic Hamiltonian matrix, spending however sometime discussing the limiting cases. We consider first the case of total spin $s = 1/2$. The two quarks q_1 and q_2 can be coupled to colour $\bar{\mathbf{3}}$ or $\mathbf{6}$, to spin 0 or spin 1. Together with the antiquark $\bar{q}_3$, spin and colour couplings are such that the cluster carries total colour 3 and spin 1/2. It follows that the space on which the Hamiltonian (Eq. (1)) acts over is four dimensional and a natural basis is provided with the four states:

$$\begin{aligned}
\phi_1 &= |(q_1q_2)_1^6\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle\,, \\
\phi_2 &= |(q_1q_2)_1^{\bar{3}}\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle\,, \\
\phi_3 &= |(q_1q_2)_0^6\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle\,, \\
\phi_4 &= |(q_1q_2)_0^{\bar{3}}\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle\,.
\end{aligned} \tag{19}$$

For completeness, let us recall the following product decompositions of $SU(3)$ representations:

$$3 \times 3 = \bar{3} + 6\ ; \quad \bar{3} \times \bar{3} = 3 + \bar{6}\ ; \quad 6 \times \bar{3} = 3 + 15\,. \tag{20}$$

The states ϕ_1 and ϕ_4 have two quarks which are coupled symmetrically in colour-spin and are therefore belonging to the $(6 \times 6)_S = 21$-dimensional representation of $SU(6)_{cs}$, the states ϕ_2 and ϕ_3 are antisymmetric in colour-spin of the two quarks and fall in the $(6 \times 6)_A = 15$-dimensional representation.

Note that if the two quarks are identical in flavour, the states ϕ_1 and ϕ_4 vanish due to the Pauli principle.

The complete expression for the colour magnetic Hamiltonian H_{CM} acting on the 4-dim vector $\overrightarrow{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ reads

$$H_{CM} =$$

$$-\begin{bmatrix}
\frac{4}{3}C_{12}+\frac{20}{3}(C_{13}+C_{23}) & 4\sqrt{2}\,(C_{13}-C_{23}) & \frac{10}{\sqrt{3}}\,(C_{13}-C_{23}) & 2\sqrt{6}\,(C_{13}+C_{23}) \\
4\sqrt{2}\,(C_{13}-C_{23}) & -\frac{8}{3}C_{12}+\frac{8}{3}(C_{13}+C_{23}) & 2\sqrt{6}\,(C_{13}+C_{23}) & \frac{4}{\sqrt{3}}\,(C_{13}-C_{23}) \\
\frac{10}{\sqrt{3}}\,(C_{13}-C_{23}) & 2\sqrt{6}\,(C_{13}+C_{23}) & -4\,C_{12} & 0 \\
2\sqrt{6}\,(C_{13}+C_{23}) & \frac{4}{\sqrt{3}}\,(C_{13}-C_{23}) & 0 & 8\,C_{12}
\end{bmatrix}. \tag{21}$$

It is easily seen from this matrix that, if we impose flavour symmetry for the two quarks ($C_{12} = C_{23}$), we get a matrix operating over two invariant subspaces $\{\phi_1, \phi_4\}$ and $\{\phi_2, \phi_3\}$, respectively. If, in addition we impose full flavour symmetry for the interaction and assume that the qq and $q\bar{q}$ interactions are the same (so that $C_{ij} = C$), then we have the matrix

$$H_{\rm CM} = -C \cdot \begin{bmatrix} \frac{44}{3} & 0 & 0 & 4\sqrt{6} \\ 0 & \frac{8}{3} & 4\sqrt{6} & 0 \\ 0 & 4\sqrt{6} & -4 & 0 \\ 4\sqrt{6} & 0 & 0 & 8 \end{bmatrix} \tag{22}$$

and we fall back on the old results [14, 18] where the eigenvalues of the colourmagnetic interaction are - 21.88C and - 0.98C for the case when the two quarks are coupled symmetrically in colour-spin. For antisymmetric colour spin the eigenvalues are $-9.68C$ and $+11.02C$.

In no case are the eigenvectors corresponding to sharp values of the total colour-spin. They are mixtures of the 6- and 120-dimensional representations as well as of the 6- and $\overline{84}$-representations of the colour spin $SU(6)_{cs}$ algebra when considering the $(qq\overline{q})^3_{1/2}$ system. Indeed, performing the product of $SU(6)$ representations:

$$21 \times \overline{6} = 6 + 120 \quad \text{and} \quad 15 \times \overline{6} = 6 + \overline{84}\,, \tag{23}$$

and examining the corresponding $SU(3) \times SU(2)$ decompositions:

$$6 = (3\,, \frac{1}{2}) \quad 120 = (3 + 15\,, \frac{1}{2} + \frac{3}{2}) + (\overline{6}\,, \frac{1}{2}) \quad \overline{84} = (15\,, \frac{1}{2}) + (3 + \overline{6}\,, \frac{1}{2} + \frac{3}{2})\,, \tag{24}$$

one easily sees that both the 6- and 120-$SU(6)$ representations contain a triplet of colour and doublet of spin, and that is also the case for the couple of representations 6 and $\overline{84}$.

Moreover, if we decouple the antiquark (going to the heavy-quark limit or considering relative spatial wave functions that have no s-wave overlap) putting $C_{13} = C_{23} = 0$, the effective Hamiltonian $H_{\rm CM}$ is diagonal, with elements which are the well known colour-magnetic energies for colour sextet and triplet diquarks.

As has been remarked before, if the two quarks are identical in flavour, the matrix is 2×2 and the states ϕ_1 and ϕ_4 disappear.

In the flavour symmetry limit, the states ϕ_1 and ϕ_4, which have the two quarks in the symmetric colour spin representation 21, are associated with the flavour $SU(3)$ representation $f = \overline{3}$, while the states ϕ_2 and ϕ_3 stand in the $f = 6$ representation as the two quarks are in the antisymmetric representation of colour spin.

Note that the flavour content $(qq\overline{q})$ is $\overline{3}\times\overline{3} = 3+\overline{6}$ for ϕ_1 and ϕ_4 and $6\times\overline{3} = 3+15$ for ϕ_2 and ϕ_3.

When the triquark $(qq\bar{q})$ is combined with the (most strongly bound) diquark (qq) which has $c = \bar{3}, s = 0$ and flavour $f = \bar{3}$, the total $(qqqq\bar{q})$ states containing ϕ_1 and ϕ_4 will be in the flavour representation $(3 + \bar{6}) \times \bar{3} = 1 + 8 + 8 + \overline{10}$, while the states containing ϕ_3 and ϕ_4 will be in the $(3 + 15) \times \bar{3} = 1 + 8 + 8 + 10 + 27$ flavour representations.

The representations $\overline{10}$ in the first group, and 27 in the second group, manifestly contain exotics.

As we have seen, ϕ_1 and ϕ_4 will mix as well as ϕ_2 and ϕ_3 if there is colour-magnetic interaction ($C_{q\bar{q}} \neq 0$) between the antiquark and the quarks. When flavour symmetry is broken, all states will generally mix; this corresponds to mixing of states in different flavour representations.

If we use isospin symmetric u and d quarks, then $C_{13} = C_{23}$ and some states with different flavour symmetry will not mix. This is the case for all models of the exotic Θ^+ which is assumed to be $(ud\, ud\bar{s})$, and it will only belong to the $f = \overline{10}$ representation.

We conclude with the total spin 3/2 case. Then the matrix representation of $H_{\rm CM}$ is acting over the space

$$\begin{aligned} \pi_1 &= |(q_1 q_2)_1^6\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle \,, \\ \pi_2 &= |(q_1 q_2)_1^{\bar{3}}\rangle \otimes |(\bar{q}_3)_{1/2}^{\bar{3}}\rangle \,, \end{aligned} \tag{25}$$

and reads

$$H_{\rm CM} = - \begin{bmatrix} \frac{4}{3}\, C_{12} - \frac{10}{3}(C_{13} + C_{23}) & -2\sqrt{3}\,(C_{13} - C_{23}) \\ -2\sqrt{3}\,(C_{13} - C_{23}) & -\frac{8}{3}\, C_{12} - \frac{4}{3}\,(C_{13} + C_{23}) \end{bmatrix}. \tag{26}$$

5. *The* $(qq\bar{q})_s^{\bar{6}}$ *triquark*

This is the second type of triquark (the first one being the colour octet one) to which we devoted a special attention [16] for constructing possibly narrow multiquark baryons made of two clusters. In contrast with the previous case where the cluster carries colour $\bar{3}$, we have here a system where there is <u>no</u> combination of $(q\bar{q})$ that are invariant under $SU(3)_c$ transformations. Evidently all $(q\bar{q})$ must be in colour octets in order to couple with a colour triplet and provide a colour $\bar{6}$. This triquark (and the ones in the following sections) is therefore protected from dissociation into a quark and a colour singlet meson.

The space over which $H_{\rm CM}$ acts is two dimensional, and as basic states we choose

$$\begin{aligned} \psi_1 &= |(q_1q_2)_1^{\overline{3}}\rangle \otimes |(\overline{q}_3)_{1/2}^{\overline{3}}\rangle\,, \\ \psi_2 &= |(q_1q_2)_0^{\overline{3}}\rangle \otimes |(\overline{q}_3)_{1/2}^{\overline{3}}\rangle\,. \end{aligned} \tag{27}$$

It is understood here that colour $= \overline{6}$ and spin is 1/2 for the $(qq\overline{q})$ system.

It should be noted that we have again chosen a basis where states are not eigenstates for the total colour-spin and that the state ψ_2 is not present if the two quarks have identical flavour. For the state ψ_1, colour-spin is a mixture of 6 and $\overline{84}$ (with flavour 3 and 15, respectively), while for the state ψ_2, the colour-spin is a mixture of 6 and 120 (with flavour 3 and $\overline{6}$).

Over this basis the Hamiltonian reads

$$H_{\rm CM} = -\begin{bmatrix} -\frac{8}{3}\,C_{12} - \frac{4}{3}(C_{13}+C_{23}) & -2\sqrt{3}\,(C_{13}-C_{23}) \\ -2\sqrt{3}\,(C_{13}-C_{23}) & 8\,C_{12} \end{bmatrix}. \tag{28}$$

In the flavour-symmetric case the Hamiltonian is diagonal.

When this triquark $(qq\overline{q})$ is combined with the (most strongly bound) diquark (qq) which has $c = \overline{6}, s = 1$ and flavour $f = \overline{3}$, the total $(qqqq\overline{q})$ states containing ψ_1 stand in the flavour representation $(3+15)\times\overline{3} = 1+8+10+27$, while the states containing ψ_2 stand in the $(3+\overline{6})\times\overline{3} = 1+8+8+\overline{10}$ flavour representations.

We add that if the spin of this cluster is 3/2, the only state is

$$\psi_3 \;=\; |(q_1q_2)_1^{\overline{3}}\rangle \otimes |(\overline{q}_3)_{1/2}^{\overline{3}}\rangle\,.$$

It is a pure state in the $\overline{84}$ of $SU6_{cs}$, and the eigenvalue of $H_{\rm CM}$ is $-\frac{8}{3}\,C_{12}+\frac{2}{3}(C_{13}+C_{23})$.

6. *The* $(qq\overline{q})_s^{15}$ *"triquark"*

For this last colour configuration, basis states for a total spin 1/2 cluster can be chosen as follows:

$$\begin{aligned} \mu_1 &= |(q_1q_2)_1^{6}\rangle \otimes |(\overline{q}_3)_{1/2}^{\overline{3}}\rangle\,, \\ \mu_2 &= |(q_1q_2)_0^{6}\rangle \otimes |(\overline{q}_3)_{1/2}^{\overline{3}}\rangle\,, \end{aligned}$$

leading to the Hamiltonian

$$H_{\rm CM} = -\begin{bmatrix} \frac{4}{3}\,(C_{12}-C_{13}-C_{23}) & -\sqrt{3}\,(C_{13}-C_{23}) \\ -\sqrt{3}\,(C_{13}-C_{23}) & -4\,C_{12} \end{bmatrix}. \tag{29}$$

Finally, if the total spin is 3/2, then only one state survives, and the corresponding eigenvalue of $H_{\rm CM}$ reads $\frac{4}{3}\,C_{12} + \frac{2}{3}(C_{13} + C_{23})$.

7. Conclusion

We have presented a detailed computation of the colour-magnetic Hamiltonian for s-wave triquark clusters in the case of flavour symmetry breaking. Such a study appears to us of some importance at this time when the possible existence of exotics hadrons is considered again. Indeed, the two main properties of such possible states being their low masses and their narrow widths, a precise evaluation by the theory of their masses becomes essential. Let us emphasize once more that the expressions which are provided can be used also for the simple isospin breaking as for triquark cluster containing simultaneously light and heavy quarks.

As we have noted before [11], it is awkward to explain the low mass of todays experimental exotic pentaquark signals with colour-magnetic interactions only. But whatever the dynamics can be, the colour-magnetic interaction must play a role.

One may wonder to what extent the above results could be used to easily determine the expression of $H_{\rm CM}$ with flavour symmetry breaking for clusters with more than three quarks. Let us answer in part to this question by considering the case of colour singlet, s-wave $(qq\overline{qq})$ states: then, using a simple argument, one can deduce that a simple substitution in the matrix $H_{\rm CM}$ of Eq.(21) provides the desired result.

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GROZDOVI OBOJENIH S-VALNIH KVARKOVA S LOMLJENJEM OKUSNE SIMETRIJE

Proučavamo svojstva obojenih tročestičnih s-valnih kvarkovskih grozdova s lomljenjem okusne simetrije. Kratko se spominje važnost tih grozdova za pentakvarkovske modele.