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LETTER TO THE EDITOR

NON-TRANSITIVE QUANTUM GAMES

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Dedicated to the memory of Professor Dubravko Tadić

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We present two alternative approaches to constructing 3×3 entangled quantum games, based on different formulations of mixed strategies in a quantum game. Although these formulations are quite similar in 2×2 games (2 players $\times$ 2 choices), their differences become pronounced in the 3×3 case. A 3×3 classical game is the simplest platform which allows for non-transitive strategies A, B, C, where A beats B, B beats C, and C beats A ($A > B > C > A$). We consider non-transitive strategies in both formulations of 3×3 quantum games, and show that non-transitivity survives in the quantum versions of the corresponding classical games. Some physical implications of these results are also considered.

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Recent years have witnessed a rapid growth of interest in quantum game theory, motivated in part by potential applications to quantum computing. Quantum games are generally derived from the corresponding classical games by introducing some inherently quantum mechanical feature (such as superposition of states [1] or entanglement [2]), which can be incorporated in more than one way. In a 2-player, symmetric game, where each player has two pure strategies available, a widely discussed scheme for entanglement is that due to Eisert, Wilkins and Lewenstein (EWL) [3]. Although the EWL analysis restricted players to a subset of physically possible actions, such restrictions exist in any physical game, and their methodology is trivially generalizable. The object of the present paper is to discuss 2-player games with three pure strategies (A, B, C) using the EWL formalism for entanglement. The novel feature of such classical games is that they allow for the possibility of non-transitive strategies ($A > B > C > A$) where $A > B$ means "A beats B". It is well known that non-transitivity arises in classical games [4,5] and in real-world applications, and can lead to surprising – and seemingly paradoxical – outcomes. This naturally leads to the question of whether similar non-transitive effects arise in quantum versions of classical games and, if so, whether similarly unexpected effects can be present at the quantum level.

A simple 2-player, 3-strategy game is the children's choosing game "rock (R), scissors (S), paper (P)" – denoted by RSP – in which rock beats scissors, scissors beats paper, and paper beats rock ($R > S > P > R$). A payoff matrix for the 3 strategies of this zero-sum game is shown in Table I (quantum games with 3×3 payoff matrices and larger have have been discussed by Wang et al. [6]). We note that since RSP is a zero-sum game, it has no pure strategy Nash equilibrium. The mixed strategy Nash equilibrium (where both players use each of R, S and P with probability 1/3) produces zero expected payoffs for both players in the classical game. Since EWL showed that quantum versions of classical games can affect Nash equilibria (when players are allowed particular parameters), the question naturally arises as to whether this can happen in non-transitive quantum games. In what follows, we analyze two formulations of incorporating mixed strategies into a quantum version of RSP and show that both retain the essential non-transitive character of the classical game.

TABLE 1. In the zero-sum game of "rock, scissors, paper", a player can win regardless of the strategy chosen by an opponent. The first number in each entry of the Table is Alice's payoff and the second is Bob's. Winning strategies are non-transitive in that $R > S > P > R$.

		Bob		
		R	S	P
	R	$(0,0)$	$(1,-1)$	$(-1,1)$
Alice	S	$(-1,1)$	$(0,0)$	$(1,-1)$
	P	$(1,-1)$	$(-1,1)$	$(0,0)$

We can formulate a quantum analog of this game by using the EWL entanglement formalism. In the quantum version of this game, the strategies R, S, and P are represented by three matrices (given by U_1, U_2 and U_3 in Eq. (9) below) which act on 3-component qutrits from which payoffs are determined. The quantum game can be played as follows: In the absence of entanglement, each player (Alice and Bob) is given a qutrit, and the initial state of the pair of qutrits is symmetric under the interchange of the qutrits, which are orthogonal and denoted by $|1\rangle$, $|0\rangle$ or $|-1\rangle$. The initial state (defined as $|11\rangle = (100)_a \otimes (100)_b$) is assumed to be symmetric to ensure that the game itself is symmetric under the interchange of the players. We note that the choice of another initial state such as $|-1\,-1\rangle$ merely serves to redefine R, S and P, and hence the essential non-transitivity in this game is unaffected. Without loss of generality, we can thus assume that the same initial qutrit state is used in every game. This qutrit can then be manipulated by any of three matrices denoted by R, S and P, each of which rotates the initial state into one of the three orthogonal states. Once both players have implemented their strategies (R, S or P), the final qutrits for both players are compared using the payoff matrix in Table I. An examination of this Table shows that $R > S > P > R$. In the EWL formalism, the qutrits are entangled by an operator, J, before the players act on them with R, S or P, and then disentangled by $J^\dagger$ afterwards. The output state which determines the payoffs is then given by

$$|\psi_{\rm out}\rangle = J^\dagger (U_a \otimes U_b) J \, |\psi_{\rm in}\rangle, \tag{1}$$

where U_a and U_b are the actions of Alice and Bob, and can denote any of R, S or P in our 3×3 game. Since the EWL entanglement operator commutes with any direct-product combination of R, S and P, the net effect of introducing entanglement in this manner is to produce a quantum game whose outcomes are identical to those of the corresponding classical game.

If a game is repeated many times, a competitor may elect to play any of R, S or P in each game with probabilities p_R, p_S and p_P respectively, in which case he/she is said to be playing a *mixed strategy.* In such a circumstance quantum games can be formulated in which payoff functions unique to quantum mechanics may result. Two ways of incorporating mixed strategies in quantum games can be considered, and these lead in general to different outcomes. In one approach, p_R, p_S and p_P are simply the classical probabilities of using each of the quantum operators R, S or P. By contrast, the second approach combines the operators R, S or P and the respective probability *amplitudes* into a single matrix. The difference in the two constructions can be illustrated in the 2×2 case (Alice's 2 choices $\times$ Bob's 2 choices): In the former approach, each player would have only two options ($N =$ *no-flip* and $F = flip$) for each game,

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

which act on the initial game state, either (10) or (01). Assigning a classical probability p to using N and $(1-p)$ to using F would then define a mixed strategy, and this entanglement technique requires that a player make a choice in each game. It should be noted that other choices for N and F are interesting to study [7], but we shall restrict our discussion to those given in Eq. (2). A maximized entanglement operation, J, can be introduced into such a game that commutes with any direct-product combination of N and F but does not commute with a general matrix,

$$J = \mathrm{e}^{\mathrm{i}(\pi/4)F\otimes F} = \frac{1}{\sqrt{2}}\big(N \otimes N\big) + \frac{\mathrm{i}}{\sqrt{2}}\big(F \otimes F\big), \tag{3}$$

where $N \otimes N = N(\text{Alice}) \otimes N(\text{Bob})$, etc.. Payoff functions for both players unique to quantum mechanics are possible if either player uses matrices other than N or F.

In the second approach, the actions available to each player, along with the probability *amplitudes* for selecting them, are built into the single unitary matrix U,

$$U = \begin{pmatrix} \sqrt{p} & -\sqrt{1-p} \\ \sqrt{1-p} & \sqrt{p} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4}$$

Once $p = p(\theta)$ has been chosen, a game can be played many times without the player making any future decisions. An entanglement matrix for such a game was provided by EWL,

$$J = \mathrm{e}^{\mathrm{i}(\pi/4)F'\otimes F'} = \frac{1}{\sqrt{2}}\big(N \otimes N\big) + \frac{\mathrm{i}}{\sqrt{2}}\big(F' \otimes F'\big), \tag{5}$$

where

$$F' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

In the 2×2 case, the entanglement matrices in Eqs. (3) and (5), which correspond to different formulations of mixed strategies, are structurally similar. However, in a 3×3 game, the equivalent entanglement operations clearly reflect their different implementations of a mixed strategy.

Before turning to the 3×3 case, we note that although some properties of quantum games are identical to those of classical games [8], the EWL entanglement technique can lead to payoff functions that cannot be reproduced in a classical game. Consider a situation where each player introduces an additional phase, ϕ, to his/her play,

$$U(\theta,\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi}\cos\theta & -\sin\theta \\ \sin\theta & \mathrm{e}^{-\mathrm{i}\phi}\cos\theta \end{pmatrix}. \tag{7}$$

It follows from Eqs. (1) and (7) that the final expected payoff function, $\overline{\$}_a$, for Alice is

$$\begin{aligned} \overline{\$}_a \quad = \quad & a_{11}c_a^2c_b^2\cos^2(\phi_a+\phi_b) + a_{10}\big(c_as_b\cos\phi_a - s_ac_b\sin\phi_b\big)^2 \\ & + a_{01}\big(s_ac_b\cos\phi_b - c_as_b\sin\phi_a\big)^2 + a_{00}\big[s_as_b + c_ac_b\sin(\phi_a+\phi_b)\big]^2, \end{aligned} \tag{8}$$

where ϕ_a (ϕ_b) is the additional phase for Alice (Bob), $c_a = \cos\theta_a$, $s_a = \sin\theta_a$ and the a's are Alice's payoff coeffcients. For non-zero but fixed values of ϕ_a and ϕ_b, Alice's payoff is non-linear in $p_a = c_a^2$, whereas a classical 2×2 game always produces only payoff functions that are linear in p_a.

The 2×2 EWL entanglement formalism can be shown to work equally well with either formulation of a mixed strategy. However, in the 3×3 case (2 players with 3 choices each), these approaches require different entanglement matrices as we now discuss. We begin by exhibiting the 3×3 analog of the operators N and F in Eq. (2) which we take to be

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{9}$$

where $[U_2, U_3] = 0$. The players use these matrices in different spaces to act on initial wave functions such as $|11\rangle = (100)_a \otimes (100)_b$. The analog of the first approach for constructing a mixed strategy is for each player to assign classical probabilities p_i to each of the U_i, where $\sum p_i = 1$. (Other choices for these matrices lead to identical results.) The commutativity of U_1, U_2 and U_3 can be exploited to create a mixing matrix similar to that of EWL [3]. The matrix

$$J = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_{ij} U_i \otimes U_j \tag{10}$$

can be used if the α_{ij} are restricted such that J is unitary. Since the set of $U_i \otimes U_j$ form a group, real solutions for the coefficients are easy to find if the following simplifying assumptions are made. First, we assume all of the α_{ij} are equal to a constant b, except for one which is equal to another constant a. In addition, we can choose to have the direct product matrices with non-zero α_{ij} form a closed subgroup. If J is then taken to have the form

$$J = b \sum_i \sum_j \big(U_i \otimes U_j\big) + (a-b)\big(U_k \otimes U_m\big), \tag{11}$$

where the sums are over members of the subgroup, and (k, m) denote the direct product matrix chosen to have the coefficient a, then its inverse is

$$J^\dagger = b \sum_i \sum_j \big(U_i^\dagger \otimes U_j^\dagger\big) + (a-b)\big(U_k^\dagger \otimes U_m^\dagger\big). \tag{12}$$

This is equivalent to

$$J^\dagger = b \sum_i \sum_j \big(U_i \otimes U_j\big) + (a-b)\big(U_k^\dagger \otimes U_m^\dagger\big). \tag{13}$$

The constraints on the α_{ij} implied by unitarity can be analyzed by calculating $J^\dagger J$,

$$\begin{aligned} J^\dagger J = b^2 \sum_i \sum_j \big(U_i \otimes U_j\big) G + b(a-b) \sum_i \sum_j \big(U_i \otimes U_j\big)\big(U_k \otimes U_m\big) \\ + b(a-b) \sum_i \sum_j \big(U_i \otimes U_j\big)\big(U_k^\dagger \otimes U_m^\dagger\big) + (a-b)^2 \mathbf{1} \otimes \mathbf{1}, \end{aligned} \tag{14}$$

where $\mathbf{1}$ is the unit matrix and G is the number of subgroup members. For example, if $k = m = 1$ then

$$J^\dagger J = \sum_i \sum_j \big(U_i \otimes U_j\big)\big[b^2 G + 2b(a-b)\big] + (a-b)^2 \mathbf{1}\otimes\mathbf{1}, \tag{15}$$

and unitarity demands

$$1 = (a-b)^2, \tag{16}$$

$$0 = b^2 G + 2b(a-b). \tag{17}$$

The solution to these equations is ($a = -1 + 2/G,\ b = 2/G$). Note that although several subgroups may be used (e.g. $U_1 \otimes U_1,\ U_2 \otimes U_3,\ U_3 \otimes U_2$) to construct an entanglement matrix, only the set ($U_1 \otimes U_1,\ U_2 \otimes U_2,\ U_3 \otimes U_3$) is the direct analog of Eq. (3),

$$J = -\frac{1}{3}U_1 \otimes U_1 + \frac{2}{3}U_2 \otimes U_2 + \frac{2}{3}U_3 \otimes U_3 \tag{18}$$

$$= \begin{pmatrix} -1/3 & 0 & 0 & 0 & 2/3 & 0 & 0 & 0 & 2/3 \\ 0 & -1/3 & 0 & 0 & 0 & 2/3 & 2/3 & 0 & 0 \\ 0 & 0 & -1/3 & 2/3 & 0 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 2/3 & -1/3 & 0 & 0 & 0 & 2/3 & 0 \\ 2/3 & 0 & 0 & 0 & -1/3 & 0 & 0 & 0 & 2/3 \\ 0 & 2/3 & 0 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 2/3 & 0 & 0 & 0 & -1/3 & 0 \\ 2/3 & 0 & 0 & 0 & 2/3 & 0 & 0 & 0 & -1/3 \end{pmatrix},$$

where $U_1 \otimes U_1 \equiv U_1(\text{Alice}) \otimes U_1(\text{Bob})$, etc. The entanglement matrix in Eq. (18) is constructed to be used in a game where players randomly select an action for each game based on classical probabilities associated with a defined mixed strategy. This matrix entangles the qutrits while still obeying the appropriate commutation relations,

$$0 = [J, U_i \otimes U_j] \qquad \forall\ i, j\,. \tag{19}$$

Unlike the 2×2 case, an initial eigenstate does not become fully entangled by this matrix: the initial state $|11\rangle$, for example, is transformed into

$$J|11\rangle = -\frac{1}{3}|11\rangle + \frac{2}{3}|00\rangle + \frac{2}{3}|-1\,-1\rangle. \tag{20}$$

If each amplitude is denoted as c_i, then the degree of entanglement (as measured by the von Neumann entropy [9,10]), E, is

$$E = -\sum_i |c_i|^2 \log_3 |c_i|^2 = 0.88\,, \tag{21}$$

where log base 3 is used to ensure that maximum entanglement corresponds to $E = 1$.

A generic entanglement matrix can be constructed for an $N \otimes N$ game. Let $U_1, U_2, \ldots, U_N$ ($U_i U_j = U_{i+j}$, $U_i^\dagger = U_{-i}$ and $U_{N+1} = U_1$) be commuting and orthogonal matrices, each capable of directly transforming an initial state into an eigenstate of some specified Hamiltonian. The elements of each matrix are taken to be either 0 or 1. If J is assumed to have the form

$$J = \sum_{i=1}^{N} \alpha_i U_i \otimes U_i, \tag{22}$$

then

$$JJ^\dagger = \sum_{i=1}^{N} \alpha_i U_i \otimes U_i \sum_{j=1}^{N} \alpha_j^* U_j^\dagger \otimes U_j^\dagger\,, \tag{23}$$

$$= \mathbf{1} \otimes \mathbf{1} \sum_{k=1}^{N} |\alpha_k|^2 + \sum_{i=1}^{N} \sum_{j=1 (j \neq i)}^{N} \alpha_i \alpha_j^* U_i U_j^\dagger \otimes U_i U_j^\dagger\,. \tag{24}$$

Unitarity requires that

$$1 = \sum_{k=1}^{N} |\alpha_k|^2, \tag{25}$$

and also that

$$0 = \sum_{i=1}^{N} \sum_{j=0}^{N-1} \alpha_i \alpha_j^* U_i U_j^\dagger \otimes U_i U_j^\dagger. \tag{26}$$

If $s \equiv i - j$, then the second unitarity condition becomes

$$0 = \sum_{j=1}^{N} \alpha_{s+j} \alpha_j^* U_s \otimes U_s \qquad \forall\, s[1, N-1], \tag{27}$$

$$= U_s \otimes U_s \sum_{j=1}^{N} \alpha_{s+j} \alpha_j^* \qquad \forall\, s[1, N-1], \tag{28}$$

or

$$0 = \sum_{j=1}^{N} \alpha_{s+j} \alpha_j^* \qquad \forall\ s[1, N-1]. \tag{29}$$

For the $N = 3$ case, $\alpha_1 = -1/3$, $\alpha_2 = \alpha_3 = 2/3$ is one solution to the unitarity requirements. For the general $N \times N$ case, the combination of Eqs. (25) and (29) thus leads to an expression for the α_i in Eq. (22), and hence to an appropriate expression for J.

We turn next to a quantum formulation of 3×3 games analogous to that following from Eq. (4). This method uses quantum amplitudes as a means of defining a mixed strategy. A matrix that can be used to transform an initial eigenstate into any other eigenstate is generated by the following matrices, which are Hermitian and unitary,

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Each player can use the matrix $U(x, y)$,

$$U(x,y) = \mathrm{e}^{\mathrm{i}xH_1}\mathrm{e}^{\mathrm{i}yH_2} \tag{31}$$

$$= \begin{pmatrix} \mathrm{e}^{\mathrm{i}x}\cos y & \mathrm{i}\mathrm{e}^{\mathrm{i}x}\sin y & 0 \\ \mathrm{i}\sin y\cos x & \cos x\cos y & \mathrm{i}\mathrm{e}^{\mathrm{i}y}\sin x \\ -\sin y\sin x & \mathrm{i}\sin x\cos y & \mathrm{e}^{\mathrm{i}y}\cos x \end{pmatrix} \tag{32}$$

to transform an initial state, $|1\rangle = (100)$, into any other state to simulate a pure strategy. For this choice of $U(x, y)$, the additional phases that have been introduced have no net effect on probabilities or payoff functions. The choices of x and y define a player's mixed strategy in a 3×3 game, just as the choice of θ (Eq. (4)) does in the 2×2 game. $U(x, y)$ satisfies the commutation relation $0 = [U(x, y), F'']$, where

$$F'' = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{33}$$

Because F'' and $U(x, y)$ commute, so do $F'' \otimes F''$ and $U(x_a, y_a) \otimes U(x_b, y_b)$, where x_a and y_a (x_b and y_b) define Alice's (Bob's) mixed strategy. This suggests the introduction of the following (maximally entangled) mixing matrix,

$$\bar{J} = \mathrm{e}^{-\mathrm{i}(\pi/4)F''\otimes F''} = \frac{1}{\sqrt{2}}\big(\mathbf{1}\otimes\mathbf{1} - \mathrm{i}F''\otimes F''\big), \tag{34}$$

which commutes with any $U(x_a, y_a) \otimes U(x_b, y_b)$. The matrix $\bar{J}$ in Eq. (34) is the exact 3×3 analog of the entanglement matrix used by EWL in their discussion of the "prisoner's dilemma" [3]. There is considerable freedom on the part of each player in how he/she chooses to deviate from $U(x, y)$, since a variable equivalent to ϕ can be introduced into a player's transformation matrix in numerous ways.

To explore the concept of non-transitivity in either formulation, we consider the 3×3 game "rock, scissors, paper" (R, S, P) shown in Table I ($R > S > P > R$). As shown in the Table, a payoff of $+1$ has been assigned to winning, -1 to losing, and 0 for both in case of a tie. It is well known that in a 2×2 game such as the "prisoner's dilemma", a player using a quantum strategy can improve his/her expected payoff provided that his/her opponent continues to use a classical strategy. The question then arises whether this can also happen in a 3×3 game which is non-transitive at the classical level. In what follows, we show that games which are non-transitive at the classical level retain their non-transitive character at the quantum level in the EWL formalism. The result of this is that a player cannot guarantee a higher payoff than his/her opponent by playing a quantum version of a classical non-transitive game (even if the opponent is allowed to play classical strategies).

Since the survival of non-transitivity in the quantum domain is a fundamental feature in 3×3 games, we present a proof of this result which clarifies some of the underlying assumptions. We assume that Alice uses a strategy (which is separable into a product of 'classical' and 'quantum' operators), $Q' = U_i Q$, for every game and that Bob uses some 'classical' strategy U_j. In this case, the final game state starting from $|11\rangle$ is

$$\begin{aligned} |\psi_f\rangle &= J_b^\dagger [U_i Q \otimes U_j] J_b |11\rangle \ = \ J_b^\dagger [U_i \otimes U_j] J_b J_b^\dagger [Q \otimes \mathbf{1}] J_b |11\rangle \\ &= [U_i \otimes U_j] J_b^\dagger [Q \otimes \mathbf{1}] J_b |11\rangle. \end{aligned} \tag{35}$$

If $|\psi_{jq}\rangle \equiv J_b^\dagger [Q \otimes \mathbf{1}] J_b |11\rangle$, then the final state becomes

$$|\psi_f\rangle = U_i \otimes U_j |\psi_{jq}\rangle. \tag{36}$$

Non-transitivity survives because Bob is still able to win regardless of what Alice does: he can turn any advantage that Alice creates with Q into an advantage to himself simply by appropriately choosing another U_j. This can easily be seen by noting that the only state $|\psi_{jq}\rangle$ that can be created by Alice's action Q in which Bob cannot better his position by changing strategy is

$$\begin{aligned} |\psi_{jq}\rangle &= \alpha(|10\rangle + |1\,{-1}\rangle + |11\rangle) + \beta(|01\rangle + |0\,{-1}\rangle + |00\rangle) \\ &\quad + \gamma(|{-1}1\rangle + |{-1}0\rangle + |{-1}\,{-1}\rangle). \end{aligned} \tag{37}$$

This follows by noting that if the initial state is $|11\rangle$, then the winning final states for Alice are $R \otimes S\,|11\rangle = R_a \otimes S_a\,|11\rangle = |10\rangle$, $P \otimes R\,|11\rangle = |{-1}1\rangle$ and $S \otimes P\,|11\rangle = |0{-1}\rangle$. Similarly, her losing final states are $S \otimes R\,|11\rangle = |01\rangle$, $R \otimes P\,|11\rangle = |1\,{-1}\rangle$ and $P \otimes S\,|11\rangle = |{-1}0\rangle$. The expected payoffs for both players given the previous equation are zero, and hence the introduction of a quantum strategy by Alice does not produce a superior Nash equilibrium, compared to the classical game.

In summary, we have developed a formalism for dealing with entanglement in 3×3 quantum games, when each player adopts a mixed strategy. Although this formalism can be applied to any $N \times N$ game, our focus in this paper has been on the simplest non-transitive 3×3 game. As noted above, classical non-transitive games are of great interest, since they can lead to seemingly paradoxical results in real-world examples [4,5]. One of the central results of the present paper is that non-transitivity survives in the quantum versions of the corresponding games. This naturally raises the question of whether similar apparent paradoxes can arise in physically realizable quantum systems. Although this question cannot be answered definitively at the present time, physical systems exist which exhibit the non-transitive features of both the "voter's paradox" and the "Penney paradox" [4,5]. As we discuss elsewhere [11], tables of Clebsch–Gordan coeffcients have properties similar to those of the magic square, whose non-transitive properties underlie the "voter's paradox". It is thus possible that interesting non-transitive effects may arise in ensembles of particles with non-zero angular momenta whose behavior can be modeled by the 3×3 non-transitive games that we have presented here. Such effects could conceivably give rise to apparent anomalies in high-energy or many-body physics, although the form of any anomalies remain an open question at present.

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NEPRIJENOSNE KVANTNE IGRE

Predstavljamo dvije inačice pristupa za sastavljanje upletenih kvantnih igara, koji se zasnivaju na različitim obrascima miješanih strategija kvantne igre. Iako su ti obrasci u igrama 2×2 vrlo slični (2 igrača $\times$ 2 odabira), njihove razlike postaju istaknutije u slučaju 3×3. Klasična igra 3×3 je najjednostavnija osnova koja dozvoljava neprijenosne strategije A, B, C, gdje A nadjača B, B nadjača C, i C nadjača A ($A > B > C > A$). Razmatramo neprijenosne strategije za oba obrasca kvantnih igara 3×3, i pokazujemo kako se u kvantnim inačicama zadržava neprijenosnost odgovarajućih klasičnih igara. Razmatraju se fizičke posljedice tih ishoda.