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COSMOLOGICAL SOLUTIONS IN VARIABLE REST MASS THEORY OF GRAVITATION

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We have obtained cosmological solutions in five-dimensional space-time-mass theory of gravitation by assuming components of energy momentum tensor, pressure $p = 0$ and the role of p_4 as a cosmological constant. The behaviour of the solution is discussed for the cases in which $k = -1, 0, +1$.

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1. Introduction

Many different theories of gravity alternative to Einstein's general general theory of relativity have been proposed in which either the gravitational constant G and the rest masses of the object vary with time. Wesson [1,2] discussed the difficulties encountered by these different approaches and proposed a variable-mass theory of gravity where the mass is regarded as a geometrical coordinate in a continuum 5-dimensional (5D) space-time-mass (STM). In some sense, the 4 dimensional Einstein's theory would be embedded in it. In this Kaluza-Klein type theory, the fifth coordinate is closely related to the mass m through $x^4 = Gm/c^2$, where the gravitational constant G and the velocity of light c are true constants.

Although the addition of a fifth dimension to the usual four dimensions does not alter the numerical size of the line element for local problems, it might have noticeable consequences for cosmological problems because the x^4 coordinate grows larger relative to the space coordinates. Such a possibility leads some authors to study cosmological solutions in vacuum. Wesson [3] found a vacuum solution with a vanishing cosmological constant. Chatterjee [4] and Fukui [5] obtained solutions in which the space-time properties depend both on time and rest mass. In these solutions, the fifth coordinate has been introduced as a time-like coordinate. As this would allow the existence of closed time-like orbits in the time-mass plane, Gron [6] considered a Bianchi type-I form of the metric with a space like fifth coordinate to study the inflationary cosmology. Ma [7] interpreted the rest mass as the length of the fifth-dimension subspace. Berman and Som [8] studied the cosmological consequences of a perfect fluid and the role of the fifth component considered as a cosmological constant and obtained an infrastationary model.

In the present paper, we have generalized the work of Ma [7] and obtained the cosmological solution in a five dimensional STM theory of gravitation by assuming pressure $p = 0$ and p_4 as a cosmological constant.

2. *Field equations and solutions*

We assume that the cosmological principle could be extended to the 5D space-time-mass and choose co-moving coordinates with $u_0 = 1$ and $u^\mu = 0$ ($\mu = 1, 2, 3, 4$) for

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau}.$$

We consider the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin\theta\mathrm{d}\phi^2\right) + \mu^2(t)\mathrm{d}\psi^2, \qquad (1)$$

where $a(t)$ is a spatial scale factor, $\mu(t)$ is the mass scale factor, $k = -1, 0, +1$ and units are chosen such that $c = 1$. The energy-momentum tensor for a perfect fluid is taken in the form suggested by Gron [6]

$$T^i_j = \mathrm{diag}\big(\rho, -p, -p, -p, -p_4\big). \qquad (2)$$

We restrict ourselves to the case $p = 0$. The 5D gravitational field equations

$$G_{ij} = -8\pi G T_{ij}$$

can be written as

$$3\left(\frac{\dot a^2}{a^2} + \frac{k}{a^2} + \frac{\dot a\dot\mu}{a\mu}\right) = 8\pi G\rho, \qquad (3)$$

$$2\frac{\ddot a}{a} + \frac{\ddot\mu}{\mu} + \frac{\dot a^2}{a^2} + \frac{k}{a^2} + 2\frac{\dot a\dot\mu}{a\mu} = 0, \qquad (4)$$

$$\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2} = -\frac{p_4}{3}, \qquad (5)$$

where overhead dot represents the differentiation with respect to t. Similarly, the expansion factor θ and the scalar shear σ are given by

$$\theta = 3\frac{\dot{a}}{a} + \frac{\dot{\mu}}{\mu}, \tag{6}$$

$$\sigma^2 = \frac{3}{8}\left(\frac{\dot{a}}{a} - \frac{\dot{\mu}}{\mu}\right)^2. \tag{7}$$

The covariant energy conservation law $T^{ij}_{;j} = 0$ gives the equation

$$\dot{\rho} + \rho\left(3\frac{\dot{a}}{a} + \frac{\dot{\mu}}{\mu}\right) = \frac{3\Lambda}{8\pi G}\frac{\dot{\mu}}{\mu}, \tag{8}$$

where $\Lambda = -p_4/3$, which can also be derived from Eqs. (3)–(5). The solution of Eqs. (4) and (5) are given by

$$a^2(t) = \begin{cases} c_1 \mathrm{e}^{\sqrt{2\Lambda}\, t} + c_2 \mathrm{e}^{-\sqrt{2\Lambda}\, t} + \dfrac{k}{\Lambda}, & \Lambda > 0\,, \\ c_1 \cos\left(\sqrt{-2\Lambda}\, t\right) + c_2 \sin\left(\sqrt{-2\Lambda}\, t\right) + \dfrac{k}{\Lambda}, & \Lambda < 0\,, \end{cases} \tag{9}$$

$$\mu(t) = \begin{cases} \left(c_3 \cos\left(\sqrt{\Lambda} t\right) + c_4 \sin\left(\sqrt{\Lambda} t\right)\right) a^{-1}(t), & \Lambda > 0\,, \\ \left(c_3 \mathrm{e}^{\sqrt{-\Lambda} t} + c_4 \mathrm{e}^{-\sqrt{-\Lambda} t}\right) a^{-1}(t), & \Lambda < 0\,, \end{cases} \tag{10}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Using Eqs. (9) and (10) in Eqs. (3) and (8), we get

$$\overline{\rho} = \frac{k}{a^2} \tag{11}$$

$$+\frac{\Lambda}{\sqrt{2}\,\mu a^3} \begin{cases} \left(c_1 \mathrm{e}^{\sqrt{2\Lambda}\, t} - c_2 \mathrm{e}^{-\sqrt{2\Lambda}\, t}\right)\left(c_4 \cos\left(\sqrt{\Lambda}\, t\right) - c_3 \sin\left(\sqrt{\Lambda}\, t\right)\right), & \Lambda > 0\,, \\ \left(c_3 \mathrm{e}^{\sqrt{-\Lambda}\, t} - c_4 \mathrm{e}^{-\sqrt{-\Lambda}\, t}\right)\left(c_2 \cos\left(\sqrt{-2\Lambda}\, t\right) - c_1 \sin\left(\sqrt{-2\Lambda}\, t\right)\right), & \Lambda < 0\,, \end{cases}$$

where $\overline{\rho} = 8\pi G\rho/3$ and

$$\sigma^2 = \begin{cases} \dfrac{3\Lambda}{8}\left[\dfrac{\sqrt{2}}{a^2}\left(c_1\mathrm{e}^{\sqrt{2\Lambda}t} - c_2\mathrm{e}^{-\sqrt{2\Lambda}t}\right) - \dfrac{c_4\cos\left(\sqrt{\Lambda}t\right) - c_3\sin\left(\sqrt{\Lambda}t\right)}{c_4\sin\left(\sqrt{\Lambda}t\right) + c_3\cos\left(\sqrt{\Lambda}t\right)}\right]^2, & \Lambda > 0\,, \\ \dfrac{-3\Lambda}{8}\left[\dfrac{\sqrt{2}}{a^2}\left(c_2\cos\left(\sqrt{-2\Lambda}t\right) - c_1\sin\left(\sqrt{-2\Lambda}t\right)\right) - \dfrac{c_3\mathrm{e}^{\sqrt{-\Lambda}t} - c_4\mathrm{e}^{-\sqrt{-\Lambda}t}}{c_3\mathrm{e}^{\sqrt{-\Lambda}t} + c_4\mathrm{e}^{-\sqrt{-\Lambda}t}}\right]^2, & \Lambda < 0\,, \end{cases} \tag{12}$$

$$\theta = \begin{cases} \dfrac{\sqrt{2\Lambda}}{a^2}\left(c_1\mathrm{e}^{\sqrt{2\Lambda}t} - c_2\mathrm{e}^{-\sqrt{2\Lambda}t}\right) + \dfrac{\sqrt{\Lambda}}{a\mu}\left(c_4\cos\left(\sqrt{\Lambda}t\right) - c_3\sin\left(\sqrt{\Lambda}t\right)\right), & \Lambda > 0\,, \\ \dfrac{\sqrt{-2\Lambda}}{a^2}\left(c_2\cos\left(\sqrt{-2\Lambda}t\right) - c_1\sin\left(\sqrt{-2\Lambda}t\right) + \dfrac{\sqrt{-\Lambda}}{a\mu}\left(c_3\mathrm{e}^{\sqrt{-\Lambda}t} - c_4\mathrm{e}^{-\sqrt{-\Lambda}t}\right), & \Lambda < 0\,, \end{cases} \tag{13}$$

The constants c_1, c_2, c_3, c_4 can be determined using the initial conditions at $t = t_0$ such that $a(t_0) = \alpha$, $\dot{a}(t_0) = \beta$, $\mu(t_0) = \gamma$ and $\dot{\mu}(t_0) = \delta$. Without loss of generality, we can choose $t_0 = 0$, i.e, $a(0) = \alpha$, $\dot{a}(0) = \beta$, $\mu(0) = \gamma$ and $\dot{\mu}(0) = \delta$. Hence, in view of the dimensionless variables as

$$R(x) = a(t)/\alpha, \qquad A(x) = \mu(t)/\gamma, \qquad x = \beta t/\alpha, \tag{14}$$

we get $\dot{a}(t) = \beta R'(x)$, $\ddot{a}(t) = \beta^2 R''(x)/\alpha$, $\dot{\mu}(t) = (\beta\gamma/\alpha)A'(x)$, $\ddot{\mu}(t) = \beta^2\gamma A''(x)/\alpha^2$, where prime represents the differentiation with respect to the variable x. Thus we have the solution in the form of dimensionless variables as

$$R^2(x) = \begin{cases} c_1\mathrm{e}^{\sqrt{2\lambda}\,x} + c_2\mathrm{e}^{-\sqrt{2\lambda}\,x} + \dfrac{k}{\lambda\beta^2} & \lambda > 0\,, \\ -\dfrac{k}{\beta^2}x^2 + c_1 x + c_2 & \lambda = 0\,, \\ c_1\cos\left(\sqrt{-2\lambda}\,x\right) + c_2\sin\left(\sqrt{-2\lambda}\,x\right) + \dfrac{k}{\lambda\beta^2} & \lambda < 0\,, \end{cases} \tag{15}$$

$$A(x) = \begin{cases} \left(c_3\cos\left(\sqrt{\lambda}\,x\right) + c_4\sin\left(\sqrt{\lambda}\,x\right)\right)R^{-1}(x) & \lambda > 0\,, \\ \left(c_3 x + c_4\right)R^{-1}(x) & \lambda = 0\,, \\ \left(c_3\mathrm{e}^{\sqrt{-\lambda}\,x} + c_4\mathrm{e}^{-\sqrt{-\lambda}\,x}\right)R^{-1}(x) & \lambda < 0\,, \end{cases} \tag{16}$$

where $\lambda = \dfrac{-p_4\alpha^2}{3\beta^2} = \dfrac{\Lambda\alpha^2}{\beta^2}$.

The constants c_1, c_2, c_3, c_4 in Eqs. (15) and (16) are to be determined using the initial conditions as

$$R(0) = 1, \qquad R'(0) = 1, \qquad A(0) = 1, \qquad A'(0) = \frac{\alpha\delta}{\beta\gamma}, \tag{17}$$

when $\lambda > 0$:

$$c_1 = \frac{1}{2} + \frac{k}{2\lambda\beta^2} + \frac{1}{\sqrt{2\lambda}}, \quad c_2 = \frac{1}{2} + \frac{k}{2\lambda\beta^2} - \frac{1}{\sqrt{2\lambda}}, \quad c_3 = 1, \quad c_4 = \frac{1}{\sqrt{\lambda}}\left(1 + \frac{\alpha\delta}{\beta\gamma}\right),$$

when $\lambda = 0$:

$$c_1 = 2, \quad c_2 = 1, \quad c_3 = 1 + \frac{\alpha\delta}{\beta\gamma}, \quad c_4 = 1,$$

when $\lambda < 0$:

$$c_1 = 1 + \frac{k}{\lambda\beta^2}, \quad c_2 = \sqrt{-\frac{2}{\lambda}}, \quad c_3 = \frac{1}{2} + \frac{1}{2\sqrt{-\lambda}}\left(1 + \frac{\alpha\delta}{\beta\gamma}\right), \quad c_4 = \frac{1}{2} - \frac{1}{2\sqrt{-\lambda}}\left(1 + \frac{\alpha\delta}{\beta\gamma}\right).$$

Now, $R' = 0$ leads to the maximum of R, say $R_{\rm m}$ at $x = x_{\rm m}$, where

$$\begin{aligned} x_{\rm m} &= \frac{1}{2\sqrt{2\lambda}} \log \frac{c_2}{c_1}, \\ R_{\rm m}^2 &= 2\sqrt{c_1 c_2} + \frac{k}{\lambda\beta^2}, \end{aligned} \tag{18}$$

for $\lambda > 0$, and

$$\begin{aligned} x_{\rm m} &= \frac{1}{\sqrt{-2\lambda}} \tan^{-1} \frac{c_2}{c_1} \\ R_{\rm m}^2 &= c_1 \cos\left(\tan^{-1} \frac{c_2}{c_1}\right) + c_1 \sin\left(\tan^{-1} \frac{c_2}{c_1}\right) + \frac{k}{\lambda\beta^2}, \end{aligned} \tag{19}$$

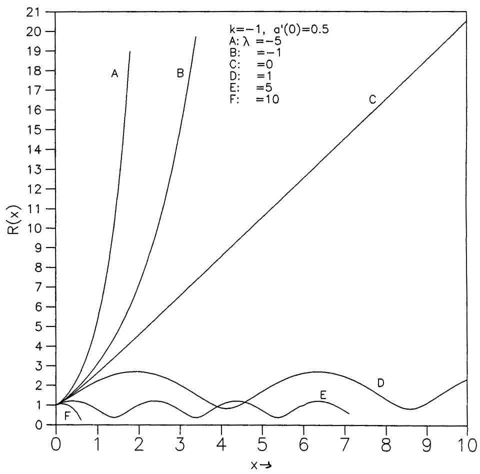
for $\lambda < 0$.

3. *Conclusion*

In this section, we have given some physical properties of the model. When the universe is spatially closed, i.e. $k = 1$, $A(x)$ contracts and $R(x)$ expands and vice-a-versa. $R(x)$ has maximum value $R_{\rm m}$ where $R_{\rm m}^2 = 2\sqrt{c_1 c_2} + k/\Lambda\beta^2$. When $c_2\, c_4 < 0$ and $c_3 c_1 < 0$, then the energy condition $\bar{\rho}(t)$ must be positive.

The ratio σ/θ tends to a finite limit as $t \to \infty$. Therefore the model is highly anisotropic for large t.

Equations (4)–(6) are integrated in view of Eqs. (14) and (17) for dimensionless variables R, A, $\overline{\rho}$ as given in Eq. (14). The behaviour of the variables R, A, $\overline{\rho}$ versus x are depicted in Figs. 1–8. Figs. 1–3 and Figs. 4–6 exhibit the behaviour of $R(x)$ and $A(x)$ respectively for different values of λ and k, where as Figs. 7 and 8 show the nature of density for $k = 0$ and $k = +1$.



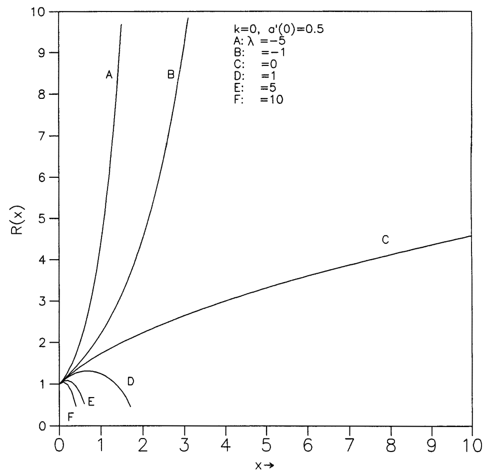
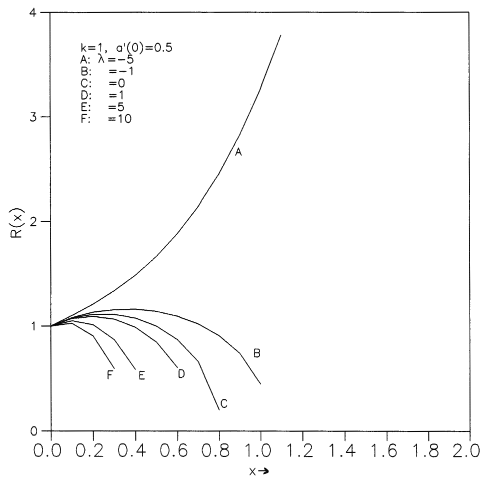


Fig. 1 (left). Behaviour of $R(x)$ for $k = -1$ and different values of λ.

Fig. 2. Behaviour of $R(x)$ for $k = 0$ and different values of λ.



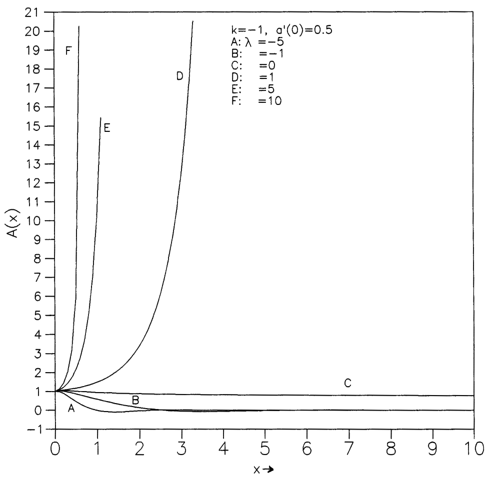
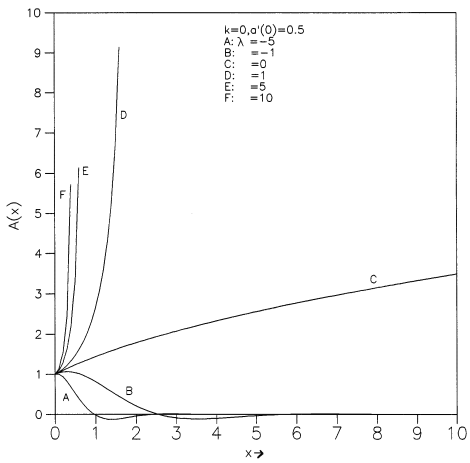


Fig. 3 (left). Behaviour of $R(x)$ for $k = 1$ and different values of λ.

Fig. 4. Behaviour of $A(x)$ for $k = -1$ and different values of λ.



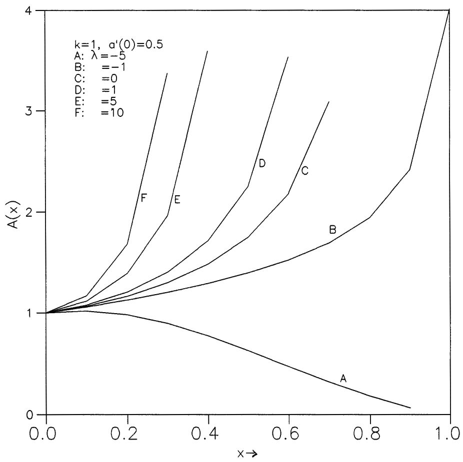
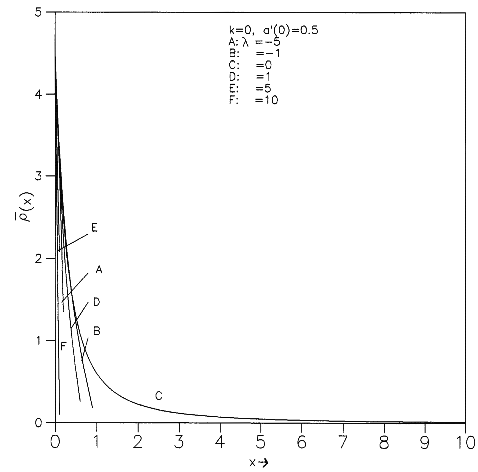


Fig. 5 (left). Behaviour of $A(x)$ for $k = 0$ and different values of λ.

Fig. 6. Behaviour of $A(x)$ for $k = 1$ and different values of λ.



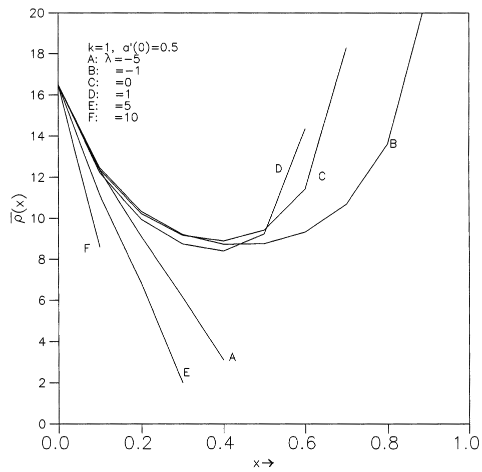


Fig. 7 (left). Behaviour of $\bar{\rho}(x)$ for $k = 0$ and different values of λ.

Fig. 8. Behaviour of $\bar{\rho}(x)$ for $k = 1$ and different values of λ.

It can be observed from the figures that as the cosmological constant λ, i.e. $-p_4$, increases, the region of existence of $R(x)$, $A(x)$ and $\overline{\rho}$ decreases for $k = -1, 0, +1$. For the same value of λ and k, one can notice that if $R(x)$ increases then $A(x)$ decreases as x increases and vice-a-versa. Figures 7 and 8 exhibit a rapid decrease of density as λ increases.

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KOZMOLOŠKA RJEŠENJA TEORIJE GRAVITACIJE S PROMJENLJIVOM MASOM MIROVANJA

Izveli smo kozmološka rješenja u petdimenzijskoj prostor-vrijeme-masa teoriji gravitacije pretpostavljajući komponente tenzora energije-impulsa, tlak $p = 0$, te uzevši p_4 u ulozi kozmološke konstante. Raspravljamo značajke rješenja za $k = +1, 0, -1$.