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CORRESPONDENCE BETWEEN TEST- AND EFFECTIVE-PARTICLE EQUATIONS OF MOTION IN RELATIVISTIC GRAVITATIONAL TWO-BODY PROBLEM

S. B. FARUQUE

Department of Physics, Shah Jalal University of Science and Technology, Sylhet 3114, Bangladesh
E-mail address: awsbf62@yahoo.com

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The correspondence between the test- and effective-particle equations of motion of a non-relativistic gravitational two-body system is well understood. But the same is not true for a relativistic two-body system. This is because the effective one-body approach to a relativistic two-body problem is not yet fully elucidated. Among the known two effective one-body approaches to relativistic two-body problem, we follow up the one addressed through a constraint Hamiltonian. We investigate the correspondence of the resulting effective one-body equation of motion with the geodetical equation of motion of a test body in the Schwarzschild field. Next, we extend the two-body problem by endowing a spin to the central body, and examine again the correspondence between the effective one-body equations of motion of such a problem with the test-body description. In particular, we show the relation between the Carters equations of geodetical motion in the Kerr field with the equations indicated by the effective one-body approach of the two-body problem. In both the Schwarzschild and the Kerr fields, we determine the location of the innermost stable circular orbit (ISCO), which is an important key for the study of astrophysical binary stars. Subsequently, we examine the correspondence between the ISCO in the test-particle orbit and in the effective-particle orbit.

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1. Introduction

The problem of relativistic motion of compact astrophysical binary systems is of great importance, as well for the ongoing as for future observations and experiments. The dynamics of a binary star is determined by the two-body problem of general relativity. This problem is not yet fully solved. Therefore, one usually applies approximation methods known as the post-Newtonian approximations to evaluate results pertaining to motion in a binary. Besides these approximations, there are two other regimes of motion described by (i) the test-mass approximation and (ii) the effective one-body description. The first case applies to the motion of, say, a planet around a star. In general relativity the motion of such a test body is termed as geodetical and the equations governing such motion are known as the geodesic equations.

The effective one-body approach to the same problem applies to the motion of bodies in a binary when the masses are comparable. In the non-relativistic mechanics, the effective particle assumes a mass $\mu = m_1 m_2/(m_1 + m_2)$ and moves around a fixed centre. The motion is of course conceptually the same as in the test particle case. One usually finds expression for the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ in the centre-of-mass (c.m.) frame defined by $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$, which gives $\boldsymbol{r}_1 = m_2\boldsymbol{r}/(m_1 + m_2)$ and $\boldsymbol{r}_2 = -m_1\boldsymbol{r}/(m_1 + m_2)$. From the solution $\boldsymbol{r} = \boldsymbol{r}(t)$ of this problem, the paths $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2 = \boldsymbol{r}_2(t)$ of the two particles separately, relative to their common centre of mass, are obtained by means of the above formulae. In non-relativistic mechanics, the problem of motion of two bodies is reduced to the problem of motion of a single body of mass μ with position given by the relative distance $\boldsymbol{r}$. This equivalent one-body problem coincides with the test-mass problem when, say, $m_2 \ll m_1$, when $\mu \simeq m_2$, $|\boldsymbol{r}_2| \simeq |\boldsymbol{r}|$ and $|\boldsymbol{r}_1| \simeq 0$.

In relativistic mechanics, the test-mass motion is described by the time-like geodesics, but the effective one-body motion has not yet been elucidated fully. In literature, there are two existing theories of effective one-body approach to the relativistic gravitational two-body problem, namely, (i) the one by Fiziev and Todorov [1], and (ii) the other by Buonanno and Damour [2]. Fiziev and Todorov [1] worked under the premise of constraint Hamiltonian mechanics formulated earlier by Dirac [3], and later on studied by Todorov [4] among many others. In this formalism, the reduced mass μ takes a relativistic form dependent on energy. Let w be the total c.m. energy of the two-body system. Then the reduced mass of the equivalent one-body, m_w, is given by

$$m_w = \frac{m_1 m_2 c^2}{w} \qquad \left(\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{for} \quad \frac{w}{c^2} \to M = m_1 + m_2\right) \tag{1.1}$$

Justification for this energy-dependent reduced mass comes from (i) the realization that if we determine the off-shell momentum square $b^2(w^2)$ for a pair of free particles as the solution for $\boldsymbol{p}^2$ of the equation

$$\frac{w}{c} = \sqrt{m_1^2 c^2 + \boldsymbol{p}^2} + \sqrt{m_2^2 c^2 + \boldsymbol{p}^2}$$

$$\Longrightarrow \boldsymbol{p}^2 = b^2(w^2) = \frac{w^4 - 2(m_1^2 + m_2^2)c^4 w^2 + (m_1^2 - m_2^2)^2 c^8}{4w^2c^2}\,, \tag{1.2}$$

we find for the effective particle c.m. energy

$$E_w = c\sqrt{m_w^2 c^2 + b^2(w^2)} = \frac{w^2 - m_1^2 c^4 - m_2^2 c^4}{2w}\,, \tag{1.3}$$

and (ii) that there are precisely three ways to factorize $b^2(w^2)$ into $(E/c - mc)$ $(E/c + mc)$ for E equal to E_1, E_2 and E_w,

$$b^2(w^2) = E_a^2/c^2 - m_a^2/c^2 = E_w^2/c^2 - m_w^2/c^2\,, \quad a = 1, 2\,. \tag{1.4}$$

This provides a fresh justification for the expressions (1.1) and (1.3) for the relativistic reduced mass and effective particle energy.

Secondly, formula (1.1) for the energy-dependent reduced mass gets justification since it appears as the coefficient to the relative velocity in the expression for the effective-particle three momentum in the c.m. frame

$$(\boldsymbol{p}_{\mathrm{eff}} =)\ \boldsymbol{p} = \boldsymbol{p}_1^{\mathrm{c.m.}} = -\boldsymbol{p}_2^{\mathrm{c.m.}} = m_1 \boldsymbol{u}_1^{\mathrm{c.m.}} = m_w \boldsymbol{u}\,, \tag{1.5}$$

where $\boldsymbol{u}$ is the relative three velocity. $\boldsymbol{u}$ together with $\epsilon = E_w/m_w c^2$ constitutes the effective-particle four-momentum in the c.m. frame.

The purpose of the present paper is to show (i) that the form of the equation of motion is the same in both the-test particle and effective-particle cases in the Schwarzschild field, (ii) that the method presented by Fiziev and Todorov for the effective one-body approach to the two-body relativistic problem can be extended to the case of a binary with one, namely, the heavier one, body spinning; it is thus equivalent to extension of the problem from the Schwarzschild to the Kerr metric case, and (iii) that a general method can be devised to find constraint Hamiltonian which is necessary to develop the equations of motion. To show explicit correspondence between the test-particle motion and effective-particle motion, we shall compute an important quantity, namely, the innermost stable circular orbit (ISCO) in both Schwarzschild and Kerr fields.

The motivation for this study comes from the fact that once the parallel between the test-particle equations of motion and effective-particle equations of motion in the Schwarzschild field is demonstrated, then the parallel between the equations we obtain for the Kerr case of effective-particle motion and those of geodetical motion found in 1968 by Carter [5] is established, and we may proceed to determine reliable dynamical properties of binary stars with one spinning central star. Hence, the analysis presented in this paper is not only of academic interest but also of observational value. The paper is organized as follows: In Sections 2 and 3, we present test-particle and effective-particle equations of motion, respectively, in the Schwarzschild field. In Sections 4 and 5, we present the test-particle and effective-particle equations of motion in the Kerr field. Finally, Section 6 contains a brief summary.

2. *Schwarzschild metric and test-particle motion*

The Schwarzschild metric is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2Gm_1}{c^2 r}\right) c^2 \mathrm{d}t^2 - \frac{1}{1 - 2Gm_1/(c^2 r)} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\,, \tag{2.1}$$

where m_1 is the mass of the source of the field. The motion of a test particle of mass m_2 in this metric is described by the geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0 \tag{2.2}$$

It can be shown that the motion is confined to the equatorial plane $\theta = \pi/2$, so that Eq. (2.2) with $\mu = 2$ (i.e. the equation for polar angle) is easily satisfied. Moreover, we can ignore Eq. (2.2) with $\mu = 1$ in favor of Eq. (2.1), which is a first integral of the geodesic equations. The remaining equations can be written as

$$\left(1 - \frac{2Gm_1}{c^2r}\right)c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \left(1 - \frac{2Gm_1}{c^2r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 1\,, \tag{2.3}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0\,, \tag{2.4}$$

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \frac{2Gm_1/c^2}{r(r - 2Gm_1/c^2)}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}t}{\mathrm{d}s} = 0\,. \tag{2.5}$$

Next, we define $v = \mathrm{d}t/\mathrm{d}s$ and $\omega = \mathrm{d}\varphi/\mathrm{d}s$. Then Eqs. (2.4) and (2.5) can be written as

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} + \frac{2}{r}\omega = 0\,, \tag{2.6}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} + \left[-\frac{1}{r} + \frac{1}{r - 2Gm_1/c^2}\right]\nu = 0\,, \tag{2.7}$$

which can be integrated to yield [6]

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{L}{r^2}\,, \tag{2.8}$$

$$\nu = \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{E}{1 - 2Gm_1/(c^2r)}\,, \tag{2.9}$$

where L and E are arbitrary constants. Substitution of these expressions in (2.3) results in

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = -1 + \frac{2Gm_1}{c^2r} + c^2E^2 - \frac{L^2}{r^2} + \frac{2Gm_1}{c^2r^3}L^2\,. \tag{2.10}$$

Using $u = 1/r$, this equation can be put in the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{2Gm_1}{c^2}u^3 - u^2 + \frac{2Gm_1}{c^2L^2}u - \frac{1 - c^2E^2}{L^2} \tag{2.11}$$

Now, it can be shown that $cL = J/m_2$, where J is the orbital angular momentum of the test body of mass m_2. Next, we define the radial distance from m_1 to m_2, r, as action, i.e., in what follows, $r = rm_2c$, where the r on the right-hand side is the actual radial coordinate expressed in length unit. Let also define $y = J/r$. Then Eq. (2.11) can be written as

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2 = \frac{2Gm_1m_2}{cJ}y^3 - y^2 + \frac{2Gm_1m_2}{cJ}y - (1-\epsilon^2)\,, \tag{2.12}$$

where $\epsilon = cE$, another constant, which later on will be identified as the energy per unit rest mass energy of the test particle. Now, let $2Gm_1m_2/(cJ) = \rho$. Then, we have

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2 = \rho y^3 - y^2 + \rho y - \beta, \qquad \beta = 1 - \epsilon^2\,. \tag{2.13}$$

Equation (2.13) is the equation of orbital motion of the test body of mass m_2 in the Schwarzschild field of the massive body of mass m_1.

It is not the purpose of the present paper to solve explicitly Eq. (2.13) for the general shape of the orbit. Interested reader will find a comprehensive account in Ref. [6]. Here, we draw attention to one important characteristic of orbits, the ISCO radius. We can find this radius by noting that the motion according to Eq. (2.13) is governed by the roots of the polynomial on the right-hand side of Eq. (2.13). Since this is a cubic polynomial, there are three roots: y_1, y_2 and y_3. By the theory of polynomials [7], those three roots are real and unequal when the term β is positive, and the discriminant of the cubic is equal to zero or less. In this case, we obtain the following conditions:

$$4\left(\frac{1}{3} - \rho^2\right)^3 \geq 27\left[\beta\rho^2 + \frac{1}{3}\left(\frac{2}{9} - \rho^2\right)\right]^2, \qquad (0<)\ \rho^2 \leq \frac{1}{3}\,. \tag{2.14}$$

When the discriminant vanishes, we get two of the roots equal, which gives rise to circular orbits. The innermost stable circular orbit (ISCO) occurs when all three roots of the polynomial become equal. This occurs when

$$\rho = \frac{1}{\sqrt{3}} \equiv y^{\mathrm{ISCO}}\,, \qquad \beta^{\mathrm{ISCO}} = \frac{1}{9}\,. \tag{2.15}$$

Solution of the first of Eqs. (2.15) gives, in units of action, the ISCO radius $r = 6Gm_1m_2/c$. Conversion to the true radius measured in units of length is achieved by dividing this by m_2c. We find

$$r^{\mathrm{ISCO}} = \frac{6Gm_1}{c^2}\,, \tag{2.16}$$

the well known ISCO radius in the Schwarzschild metric. In the next section we shall see how the effective one-body formalism modifies this result.

3. *Effective-particle motion in the Schwarzschild field*

As discussed in Sect. 1, the effective-particle approach to the relativistic two-body problem presented by Fiziev and Todorov [1] uses an energy-dependent reduced mass m_w given by Eq. (1.1). The equations of motion of this effective particle are derived from a constraint Hamiltonian. Let us briefly comment on this formalism. A world-line in a space-time manifold $\boldsymbol{M}$ is a 1-dimensional (time-like) sub-manifold $\boldsymbol{M}$, usually given by parametric equations of the type

$$x^{\mu} = x^{\mu}(t), \qquad \mu = 0,\ 1,\ 2,\ 3; \qquad -\infty < t < \infty\,. \tag{3.1}$$

For a non-relativistic system, there is a privileged choice of the evolution parameter t, namely, the time component, x^0, of the four-vector x. It does not change under homogeneous Galilean transformations (and, in general, only the origin of the time axis may be shifted). For a relativistic system, this is not the case: the separation between space and time components of x depends on the choice of the Lorentz frame (and the proper time, which is a natural evolution parameter for a single/massive/particle, and has no universal extension to many-particle systems). This makes it desirable to have a formulation of relativistic dynamics which does not depend on the choice of t [4]. This leads to the constraint Hamiltonian mechanics of Dirac [3]. It turns out that the constraint which allows to express the particle energy $E = -P_0$ as a function of its three momentum and the external field also determines the equations of motion [4].

We now turn to what Fiziev and Todorov [1] established. They have used the following constraint Hamiltonian

$$H = \frac{1}{2\lambda}(1 + g^{00}\epsilon^2 + g^{ij}u_i u_j) \approx 0\,, \tag{3.2}$$

where λ is a Lagrange multiplier which is related to the choice of an evolution parameter. In (3.2), u_i are the dimensionless three-momentum of the effective particle, in particular, the radial momentum $u_r = p_R/(m_w c)$, p_R being the actual radial momentum conjugate to the invariant distance R between the two bodies in c.m. frame. All lengths in (3.2) are expressed in units of action. In particular, the radial relative coordinate is $r \equiv R m_w c$.

Now, Fiziev and Todorov [1] used the following metric tensors:

$$g^{00} = -\frac{1}{1 - 2\alpha_G/r}\,, \tag{3.3}$$

$$g^{ij}u_i u_j = \left(1 - \frac{2\alpha_g}{r}\right)u_r^2 + \frac{J^2}{r^2}\,, \tag{3.4}$$

where $\alpha_G = Gm_1m_2/c$ is the gravitational coupling parameter which determines the effective metric and J is the orbital angular momentum of the effective particle. The resulting constraint Hamiltonian is

$$H = \frac{1}{2\lambda}\left[1 + \left(1 - \rho\frac{J}{r}\right)u_r^2 + \frac{J^2}{r^2} - \frac{\epsilon^2}{1 - \rho J/r}\right] \approx 0\,, \tag{3.5}$$

where $\rho = 2Gm_1m_2/(cJ)$ as it is in Sect. 2. The equations of motion are

$$\dot{r} = \frac{\partial H}{\partial u_r} = \left(1 - \rho\frac{J}{r}\right)\frac{u_r}{\lambda}\,, \tag{3.6}$$

$$\dot{\varphi} = \frac{\partial H}{\partial J} = \frac{J}{\lambda r^2}\,, \tag{3.7}$$

$$\dot{t} = -\frac{\partial H}{\partial \epsilon} = \frac{\epsilon}{\lambda(1-\rho J/r)}\,, \tag{3.8}$$

Now, to compare the equations in the test particle case with the effective particle case, we note that Eq. (2.10) is exactly the same as what we obtain for $\dot{r}$ from (3.6), except for the Lagrange multiplier λ and the definition of r. To be more specific, if we multiply every r in Eq. (2.10) by m_2c, and denote the resulting term by the same symbol, and multiply L by m_2 to get J, and do the corresponding adjustments, we get the following equation

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \epsilon^2 - 1 + \rho\frac{J}{r} - \frac{J^2}{r^2} + \rho\frac{J^3}{r^3}\,, \tag{3.9}$$

which is the equation we get from (3.6), except for the term λ and the definition of r which in the latter case is $r = Rm_wc \equiv rm_wc$. Equation (3.8) is similar to Eq. (2.9) in the same sense. Equation (3.7) is exactly the same as Eq. (2.8) in the above sense. To see this, we convert Eq. (2.8) as follows

$$\frac{\mathrm{d}\varphi}{m_2c\ \mathrm{d}s} = \frac{Lm_2c}{(m_2c)^2r^2} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{J}{r^2}\,, \tag{3.10}$$

where, in the second equation, all lengths are in units of action.

Now, solving (3.5) for for u_r, and substituting it in Eq. (3.6), and dividing (3.6) by (3.7), we obtain the λ-independent equation of motion

$$-\frac{\mathrm{d}y}{\mathrm{d}\varphi} = \left[\rho y^3 - y^2 + \rho y - \beta\right]^{1/2}, \tag{3.11}$$

where ρ and y are as in Eq. (2.13) except for the meaning of the terms. To clarify, we note that in the test-particle case conversion of length r into action is achieved by multiplying it by m_2c, whereas in Eq. (3.11), it is achieved via multiplication by m_wc. Moreover, J is the orbital angular momentum of the test particle in Eq. (2.13), but in Eq.(3.11) , it is the same of the effective particle. Except for these differences in definition of the terms, Eqs. (2.13) and (3.11) are identical. Now, we can identify ϵ in Eq. (2.13) as the energy per unit rest-mass energy of the test body.

Next, the effective particle ISCO is given, similarly to the previous case, by $\rho = \frac{1}{\sqrt{3}} \equiv y^{\mathrm{ISCO}}$. But, we obtain now for the radius

$$r^{\mathrm{ISCO}} = \frac{6Gm_1m_2}{c}\ \frac{1}{m_wc} = \frac{6G(m_1+m_2)}{c^2}\ \sqrt{1-2\nu(1-\epsilon)}\,, \tag{3.12}$$

where we have used

$$m_w = \frac{m_1m_2}{(m_1+m_2)}\ \frac{1}{\sqrt{1-2\nu(1-\epsilon)}}\,, \tag{3.13}$$

which follows easily from Eqs. (1.1) − (1.3) and the definitions of the various terms. Note that the deformation parameter $\nu = m_1 m_2/(m_1+m_2)^2$. Equation (3.12) gives the radial distance between the particles in the c.m. frame. This radius is not the radius which is to be compared with the radius given by the test-particle value (2.16). In fact, in the two-particle scenario, both bodies move round their common centre-of-mass. So, it is the radius vector from the centre-of-mass to the smaller mass m_2 that is to be compared with the distance between the particles in the test particle case. So, from (3.12), we obtain

$$r_2 = \frac{m_1}{m_1+m_2} r \qquad \Longrightarrow \qquad r_2^{\mathrm{ISCO}} = \frac{6Gm_1}{c^2}\sqrt{1-2\nu(1-\epsilon)}\,. \tag{3.14}$$

This radius from the center-of-mass determines the elliptic orbit of the smaller particle of mass m_2. The value (3.14) is smaller than the Schwarzschild value given by (2.16).

4. *Test particle motion in the Kerr field*

The Kerr metric is defined by

$$\mathrm{d}s^2 = -\left(1-\frac{2Gm_1 r}{c^2(r^2+a^2\cos^2\theta)}\right)c^2\mathrm{d}t^2 + \frac{r^2+a^2\cos^2\theta}{r^2-(2Gm_1 r/c^2)+a^2}\mathrm{d}r^2 + (r^2+a^2\cos^2\theta)\mathrm{d}\theta^2$$

$$+\left(r^2+a^2+\frac{2Gm_1 ra^2}{c^2(r^2+a^2\sin^2\theta)}\sin^2\theta\right)\sin^2\theta\mathrm{d}\varphi^2 - \frac{4Gm_1 ra}{c^2(r^2+a^2\cos^2\theta)}\sin^2\theta\mathrm{d}\varphi(c\mathrm{d}t)\,. \tag{4.1}$$

If we consider the spin of the central body to be very small such as $a \ll m_1$, and if we consider distances such as $m_1 < r < \infty$, we can simplify the metric (4.1), keeping in mind also that the planar motion occurs only in the equatorial plane $\theta = \pi/2$, then

$$\mathrm{d}s^2 = -\left(1-\frac{2Gm_1}{c^2 r}\right)c^2\mathrm{d}t^2 + \left(1+\frac{2Gm_1}{c^2 r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 - \frac{4Gm_1 a}{cr}\mathrm{d}\varphi\mathrm{d}t\,. \tag{4.2}$$

The equations of motion of a test particle in the Kerr field have been found by Carter [5]. For the small spin case, as we have in the metric (4.2), the equations of motion are [5,8], now with $c = 1$,

$$m_2\frac{\mathrm{d}t}{\mathrm{d}s} = -\frac{2Gm_1 a}{r(r^2-2Gm_1 r)}J + \frac{Er^2}{r^2-2Gm_1 r}\,, \tag{4.3}$$

$$m_2\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{J}{r^2} + \frac{2Gm_1 aE}{r^3(1-2Gm_1/r)}\,, \tag{4.4}$$

$$m_2\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = E^2 - m_2^2\left(1-\frac{2Gm_1}{r}\right) - \left(1-\frac{2Gm_1}{r}\right)\frac{J^2}{r^2} - \frac{4Gm_1 a}{r^3}EJ\,. \quad (4.5)$$

Now, a few steps of algebra reveal that if we approximate $\mathrm{d}\varphi/\mathrm{d}s$ by L/r, since we are considering very small a, we get

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2 = \left(\rho - \frac{2\rho a\epsilon}{J}\right)y^3 - y^2 + \rho y - \beta\,, \quad (4.6)$$

where

$$\rho = \frac{2Gm_1m_2}{J}, \quad r \equiv rm_2, \quad a \equiv am_2, \quad y = \frac{J}{r}, \quad \beta = 1-\epsilon^2 \quad \text{and} \quad J = Lm_2. \quad (4.7)$$

Putting $\rho - 2\rho a\epsilon/J = \alpha$, we obtain

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\varphi}\right)^2 = \alpha y^3 - y^2 + \rho y - \beta\,. \quad (4.8)$$

Analysis of this equation in a fashion similar to what we did in Sect. 2 yields the approximate ISCO radius

$$y^{\mathrm{ISCO}} = \frac{1}{3\alpha} = \rho\,. \quad (4.9)$$

Solution to this is

$$r^{\mathrm{ISCO}} = 6Gm_1m_2 - 12Gm_1m_2\frac{a}{J}\epsilon\,. \quad (4.10)$$

However, in (4.10] r is expressed as action. To convert to true length units, we divide (4.10) by m_2 and use c in appropriate places to obtain

$$r^{\mathrm{ISCO}} = \frac{6Gm_1}{c^2} - \frac{12Gm_1}{c^2}\,\frac{a}{J}\epsilon \quad (4.11)$$

We note that in (4.11) a is in units of action (see (4.7)).

5. *Effective particle motion in Kerr field*

Some preliminary discussion on motion of an effective particle in the Kerr field can be found in Refs. [9], [10] and [11]. Here, we begin with the general prescription to obtain the constraint Hamiltonian for the Kerr metric given in (4.2) . We first construct the square of the four gradient operator corresponding to metric (4.2)

$$g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = -\frac{1}{1-2Gm_1/r}\left(\frac{\partial}{\partial t}\right)^2 + (1-2Gm_1/r)\left(\frac{\partial}{\partial r}\right)^2$$
$$+\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}\right)^2 - \frac{4Gm_1a}{r^3}\frac{1}{1-2Gm_1/r}\left(\frac{\partial}{\partial\varphi}\right)\left(\frac{\partial}{\partial t}\right) \quad (5.1)$$

The Hamiltonian constraint is

$$H = \frac{1}{2\lambda}\left[1 + g^{\mu\nu}u_\mu u_\nu\right] \approx 0\,. \tag{5.2}$$

Now, we use the transformations

$$\frac{\partial}{\partial t} \to -\epsilon, \quad \frac{\partial}{\partial r} \to u_r, \quad \frac{\partial}{\partial \varphi} \to J, \quad \text{and} \quad Gm_1 \to \alpha_G = Gm_1 m_2\,, \tag{5.3}$$

in Eq. (5.1) to obtain the Hamiltonian (5.2) explicitly as

$$H = \frac{1}{2\lambda}\left[1 - \frac{\epsilon^2}{1 - \dfrac{2Gm_1m_2}{r}} + \left(1 - \frac{2Gm_1m_2}{r}\right)u_r^2 + \frac{J^2}{r^2} + \frac{4Gm_1m_2a}{r^3}\,\frac{1}{1 - \dfrac{2Gm_1m_2}{r}}J\epsilon\right]. \tag{5.4}$$

Note that all length parameters in (5.4) are in action units, hence, $a \equiv am_w$ and $r \equiv rm_w$. The equations of motion are

$$\dot{r} = \frac{\partial H}{\partial u_r} = \left(1 - \frac{2Gm_1m_2}{r}\right)\frac{u_r}{\lambda}\,, \tag{5.5}$$

$$\dot{r} = \frac{\partial H}{\partial J} = \frac{1}{\lambda}\left[\frac{J}{r^2} + \frac{2Gm_1m_2a}{r^3}\,\frac{\epsilon}{1 - 2Gm_1m_2/r}\right], \tag{5.6}$$

$$\dot{t} = \frac{\partial H}{\partial \epsilon} = \frac{1}{\lambda}\left[\frac{\epsilon}{1 - 2Gm_1m_2/r} - \frac{2Gm_1m_2a}{r^3}\,\frac{J}{1 - 2Gm_1m_2/r}\right]. \tag{5.7}$$

Equations (5.5) – (5.7) are equivalent to Eqs. (4.3) – (4.5). The only differences lie in that Eqs.(5.5) – (5.7) contain λ, whereas Eqs.(4.3) – (4.5) do not have this term, and in that all lengths in Eqs.(4.3) – (4.5) are in normal length units, whereas in (5.5) – (5.7) they are in action units. To see the correspondence between equations of test particle motion and the effective particle motion, we may express all lengths in Eqs. (4.3) – (4.5) in units of action by multiplying all a and r by m_2. The resulting equations resemble exactly Eqs.(5.5) – (5.7), except for the term λ, and that all a and r in (5.5) – (5.7) expressed in action units by multiplying them by m_w, that is, $r \equiv rm_w$ and $a \equiv am_w$.

Now, solving Eq.(5.4) for u_r and substituting that in (5.5), then, dividing (5.5) by (5.6) and observing the appropriate approximations as outlined in the beginning of Sect. 4, we obtain the following equation of orbital motion of an effective particle of reduced mass m_w in the deformed Kerr field

$$-\frac{\mathrm{d}y}{\mathrm{d}\varphi} \simeq \left[\alpha y^3 - y^2 + \rho y - \beta\right]^{1/2}, \tag{5.8}$$

where α is the same as defined in Sect. 4. Analysis of this equation, in the same fashion as in Sect. 4, yields the approximate ISCO radius

$$y^{\rm ISCO} = \frac{1}{3\alpha} = \rho \,. \tag{5.9}$$

Solution to this is

$$r^{\rm ISCO} = 6Gm_1m_2 - 12Gm_1m_2\frac{a}{J}\epsilon \,. \tag{5.10}$$

Conversion to true units gives us

$$r^{\rm ISCO} = 6G(m_1+m_2)\sqrt{1-2\nu(1-\epsilon)} - 12G(m_1+m_2)\sqrt{1-2\nu(1-\epsilon)}\frac{a}{J}\epsilon \,, \tag{5.11}$$

where a is in units of action. Putting the c's in the appropriate places and converting to the radius of the smaller mass from the centre-of-mass, we obtain

$$r^{\rm ISCO} = \frac{6Gm_1}{c^2}\sqrt{1-2\nu(1-\epsilon)} - \frac{12Gm_1}{c^2}\sqrt{1-2\nu(1-\epsilon)}\frac{a}{J}\epsilon \,. \tag{5.12}$$

The last factor in (5.12) can be positive or negative due to the relative sense of rotation of the bodies in the binary. The second factor in (5.12) is negative when the sense of rotation of the mass m_1 is same as the sense of revolution of the mass m_2. The corresponding orbit is prograde. In the retrograde orbit, the second term in (5.12) is positive. Therefore, we see clearly that the spin-orbit interaction splits up the ISCO in two distinct orbits.

6. *Summary*

In this paper, we have investigated the correspondence between the test-particle equations of motion and effective-particle equations of motion in the Schwarzschild and the Kerr fields. Firstly, we have derived the geodesic equations in the Schwarzschild field which govern test-particle motion in the field of massive body of mass m_1. These equations are then shown to be in parallel with the equations of motion for effective particle derived in the way shown by Fiziev and Todorov [1]. The complete analogy between the two cases confirms the validity of the approach of Ref. [1]. This motivates us to extend the two-body problem by endowing a small spin to the central body, i.e., we consider motion of a body in Kerr field. Now, in Kerr field, geodetical motion is governed by the famous Carters equations [5]. We, therefore, set our mind to formulate the effective one-body approach to the two-body problem in Kerr field in such a fashion that there results a clear correspondence between the Carters equations and those of ours. Fortunately, the constraint Hamiltonian we have constructed predicts equations of motion that are in good parallel with Carters equations.

We have computed the ISCO radius both in the Schwarzschild field and the Kerr field. In both the fields we have computed test-particle ISCO and massive-particle ISCO. The equations of motion of effective particle in the Schwarzschild and the Kerr field reduce to those of test particle when the relativistic reduced mass m_w reduces to the mass m_2 of the orbiter. In conclusion, we remark that although the analysis in this paper is rather naive, the implication of the results are conceptually very illustrative.

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ODNOS JEDNADŽBI GIBANJA ISPITNE I EFEKTIVNE ČESTICE U RELATIVISTIČKOM GRAVITACIJSKOM PROBLEMU DVA TIJELA

Odnos jednadžbi gibanja ispitne i efektivne čestice u nerelativističkom problemu dva tijela dobro je poznat. To ne vrijedi za relativistički problem dva tijela. Razlog tome je što pristup efektivnog jednog tijela još nije razjašnjen. Od dvaju poznatih pristupa efektivnog jednog tijela relativističkom problemu dva tijela, mi slijedimo pristup preko uvjetovanog hamiltonijana. Ispitujemo odnos postignute jednadžbe efektivnog jednog tijela i geodetske jednadžbe gibanja ispitnog rijela u Schwarzschildovom polju. Nadalje, proširujemo problem dva tijela razmatrajući i vrtnju središnjeg tijela, i opet ispitujemo odnos jednadžbi gibanja efektivnog jednog tijela tog problema i opisa za ispitno tijelo. Posebno, pokazujemo odnos Carterovih jednadžbi za geodetsko gibanje u Kerrovom polju i jednadžbi u pristupu efektivnog jednog tijela. U Schwarzschildovom i u Kerrovom polju određujemo najmanju stabilnu kružnu stazu (ISCO) koja je važna za proučavanje astrofizičkih binarnih zvijezda, te zatim ispitujemo odnos ISCO staze ispitnog tijela i staze efektivnog tijela.