ISSN 1330–0016
CODEN FIZBE7

DYSON-SCHWINGER EQUATIONS, THE PION AND RELATED MATTERS

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Received 13 August 2003; Accepted 26 April 2004
Online 10 October 2004

We recapitulate the aspects of Dyson-Schwinger equation (DSE) studies relevant to pseudoscalar mesons: lattice confirmation of the DSE prediction that propagators are nonperturbatively dressed in the infrared; and exact results, e.g., the leptonic decay constant vanishes for every pseudoscalar meson except the pion in the chiral limit.

PACS numbers: 12.38.Aw, 14.40.Aq, 14.65.-q, 24.85.+p UDC 539.12
Keywords: dynamical chiral symmetry breaking, Dyson-Schwinger equations, pion properties

1. *Gap equation*

The vast body of pion data available provides compelling evidence that this composite particle is a Goldstone mode of the strong interaction associated with the dynamical breaking of chiral symmetry. Therefore a legitimate understanding of pion observables, including its mass, decay constants and form factors, requires that an approach possesses a well-defined and valid chiral limit. This is impossible without a detailed grasp of the connection between current- and constituent-quarks.

In QCD the running quark mass is obtained from the solution of

$$S^{-1}(p) = Z_2\,(i\gamma\cdot p + m_{\rm bare}) + \, Z_1 \int_q^\Lambda g^2 D_{\mu\nu}(p-q)\,\frac{\lambda^a}{2}\gamma_\mu\, S(q)\,\Gamma^a_\nu(q;p)\,. \qquad (1)$$

This is the Dyson-Schwinger equation (DSE) for the dressed-quark self energy or, equivalently, QCD's *gap equation*, and it is a keystone in understanding dynamical chiral symmetry breaking (DCSB) and the relation between current- and constituent-quarks. On the right-hand side of Eq. (1): $D_{\mu\nu}(k)$ is the dressed-gluon propagator; $\Gamma^a_\nu(q;p)$ is the dressed-quark-gluon vertex; $m_{\rm bare}$ is the Λ-dependent current-quark bare mass; and $\int_q^\Lambda := \int^\Lambda d^4q/(2\pi)^4$ represents a translationally-invariant regularisation of the integral, with Λ the regularisation mass-scale. In addition, $Z_{1,2}(\zeta^2,\Lambda^2)$ are the quark-gluon-vertex and quark wave function renormalisation constants, which depend on Λ and the renormalisation point, ζ, as does the mass renormalisation constant $Z_m(\zeta^2,\Lambda^2) = Z_4(\zeta^2,\Lambda^2)/Z_2(\zeta^2,\Lambda^2)$. The solution of Eq. (1) has the form

$$S^{-1}(p) = {\rm i}\gamma\cdot p\, A(p^2,\zeta^2) + B(p^2,\zeta^2) \equiv \frac{1}{Z(p^2,\zeta^2)}\left[{\rm i}\gamma\cdot p + M(p^2)\right], \qquad (2)$$

where $M(\zeta^2) \equiv m(\zeta) := m_{\rm bare}(\Lambda)/Z_m(\zeta^2,\Lambda^2)$ is the running quark mass.

The behaviour of the nonperturbative solution of QCD's gap equation is a long-standing prediction of DSE studies [1−3], and typical results are illustrated in Fig. 1. One critical feature is that so long as the kernel of the gap equation has sufficient integrated strength at infrared momenta, one obtains a nonzero running quark mass even in the chiral limit

$$Z_4(\zeta,\Lambda)\, m(\zeta) \equiv 0\,,\ \Lambda \gg \zeta\,. \qquad (3)$$

This effect is DCSB. It is impossible at any finite order of perturbation theory and apparent in Fig. 1.

The dressed-quark propagator can be calculated in lattice-regularised QCD. Results are available in the quenched truncation, and depicted in Fig. 1 are those of Ref. [5] obtained with the current-quark masses ($\zeta = 19$ GeV)

$a\, m_{\rm lattice}$	0.018	0.036	0.072
$m(\zeta)$(GeV)	0.030	0.055	0.110

(4)

The precise agreement with DSE results is not accidental. The essential agreement between lattice results and DSE predictions was highlighted in Refs. [6], but Ref. [4] pursued a different goal. Only recently has reliable information about the gap equation's kernel at infrared momenta begun to emerge, in the continuum [7] and on the lattice [8]. Reference [4] therefore employed an *Ansatz* for the infrared behaviour of the gap equation's kernel in order to demonstrate that it is possible to correlate lattice results for the gluon and quark Schwinger functions via QCD's gap equation. This required the gap equation's kernel to exhibit infrared enhancement over and above that observed in the gluon propagator alone, which could be attributed to an amplification of the dressed-quark-gluon vertex whose magnitude is consistent with that observed in quenched lattice estimates of this three-point function [9].

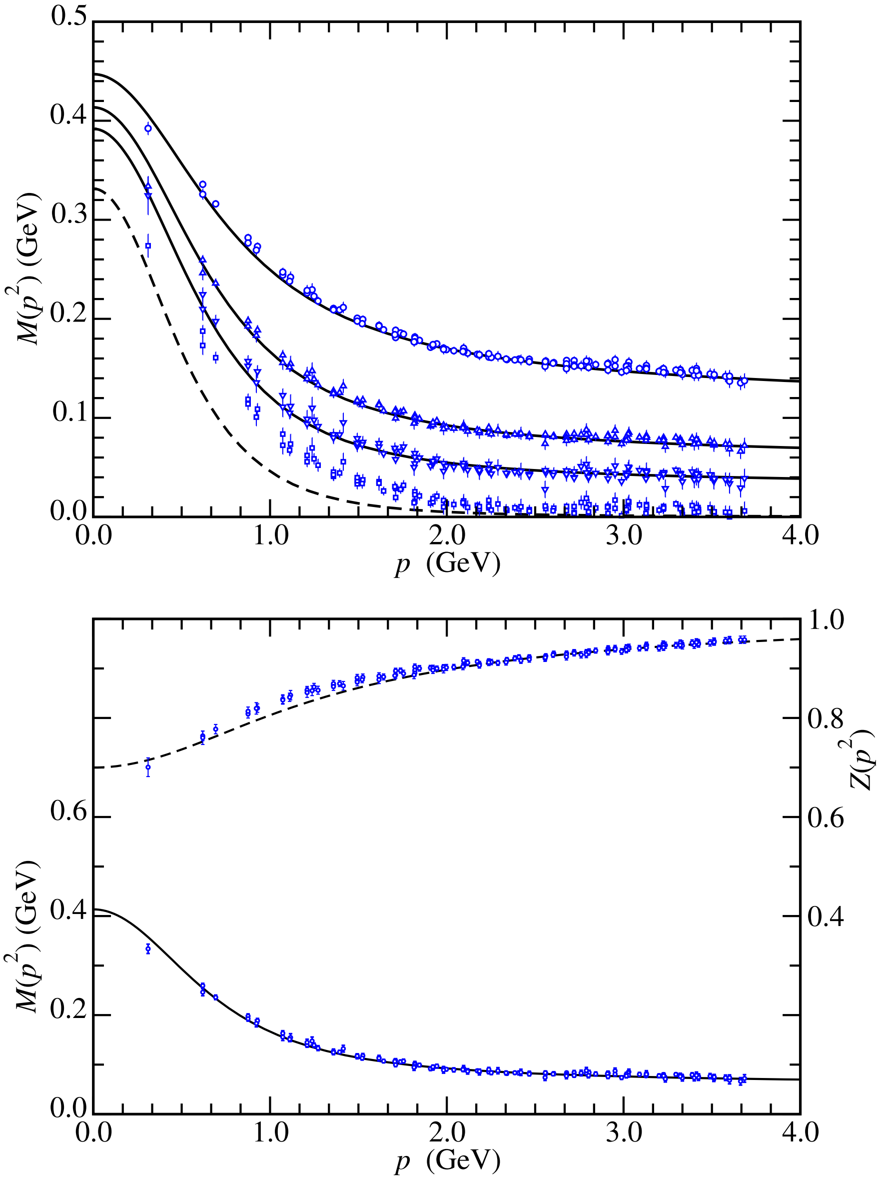

Fig. 1. Upper panel – Dashed-curve: gap equation's solution in the chiral limit; solid curves: solutions for $M(p^2)$ obtained using the current-quark masses in Eq. (4). (From Ref. [4].) Data, upper three sets: lattice results for $M(p^2)$ in GeV *at am values in Eq. (4); lower points (boxes): linear extrapolation of these results [5] to $am = 0$. Lower panel – Dashed curve, $Z(p^2)$, and solid curve, $M(p^2)$ calculated from the gap equation with $m(\zeta) = 55\,\mathrm{MeV}$ [4]. Data, quenched lattice-QCD results for $M(p^2)$ and $Z(p^2)$ obtained with $am = 0.036$ [5]. ($Z(p^2)$ is dimensionless.)*

2. *Hadrons*

It is evident that reliable knowledge of QCD's two-point functions (the propagators for QCD's elementary excitations) is available. Direct comparison with experiment requires an equally good understanding of bound states. Progress here has required the evolution of an understanding of the intimate connection between symmetries and DSE truncation schemes. The best known scheme is the weak coupling expansion, which reproduces every diagram in perturbation theory. This scheme is valuable in the analysis of large-momentum transfer phenomena because QCD is asymptotically free. However, it precludes any possibility of obtaining nonperturbative information, and bound state phenomena are intrinsically nonperturbative.

The properties of the pion are profoundly connected with DCSB, and chiral symmetry and its breaking are expressed through the axial-vector Ward-Takahashi identity ($k_\pm = k \pm P/2$, $\{\tau^j, j = 1, 2, 3\}$ are the Pauli matrices)

$$P_\mu \Gamma^j_{5\mu}(k;P) = \mathcal{S}^{-1}(k_+)\mathrm{i}\gamma_5 \frac{\tau^j}{2} + \mathrm{i}\gamma_5 \frac{\tau^j}{2}\mathcal{S}^{-1}(k_-) - \mathrm{i}\mathcal{M}(\zeta)\,\Gamma^j_5(k;P) - \Gamma^j_5(k;P)\,i\mathcal{M}(\zeta). \tag{5}$$

This identity connects the axial-vector vertex: $\Gamma^j_{5\mu}(k;P)$, P is the total momentum, with the dressed quark propagator: $\mathcal{S} = \mathrm{diag}[S_u, S_d]$, the pseudoscalar vertex: $\Gamma^j_5(k;P)$, and the current-quark mass matrix: $\mathcal{M}(\zeta) = \mathrm{diag}[m_u(\zeta), m_d(\zeta)]$. The propagator satisfies the gap equation but the vertices are determined by inhomogeneous Bethe-Salpeter equations; e.g.,

$$\left[\Gamma^j_{5\mu}(k;P)\right]_{tu} \;=\; Z_2 \left[\gamma_5 \gamma_\mu \frac{\tau^j}{2}\right]_{tu} + \int_q^\Lambda [\chi^j_{5\mu}(q;P)]_{sr}\, K^{rs}_{tu}(q,k;P)\,, \tag{6}$$

wherein $\chi^j_{5\mu}(q;P) := \mathcal{S}(q_+)\Gamma^j_{5\mu}(q;P)\mathcal{S}(q_-)$ and $K(q,k;P)$ is the dressed-quark-antiquark scattering kernel. The importance of DCSB entails that any truncation useful in understanding low energy phenomena must be nonperturbative and preserve Eq. (5), without fine tuning. This nontrivial constraint cannot be satisfied without an intimate connection between $K(q,k;P)$ and the gap equation's kernel.

One systematic truncation scheme has been identified that explicates this connection and hence preserves QCD's global symmetries [10]. It is a dressed-loop expansion of the dressed-quark-gluon vertices that appear in the half-amputated dressed-quark-antiquark scattering matrix: S^2K. The leading order term is the renormalisation-group-improved rainbow-ladder truncation, which underlies a one-parameter model of the quark-quark interaction used successfully in *ab initio* calculations of vector and flavour nonsinglet pseudoscalar meson properties [3].

2.1. *Model-independent results*

The existence of a nonperturbative, systematic and symmetry preserving truncation scheme enables exact results to be proved. For example, it is a general feature of QCD that the axial-vector and pseudoscalar vertices exhibit poles whenever $P^2 = -m_{\pi_n}^2$, where m_{π_n} is the mass of the pion or any of its radial excitations.* This can be expressed for the axial-vector vertex via

$$\begin{aligned} \Gamma_{5\mu}^j(k;P)\Big|_{P^2+m_{\pi_n}^2\approx 0} &= \frac{\tau^j}{2}\gamma_5 \left[\gamma_\mu F_R(k;P) + \gamma\cdot k k_\mu G_R(k;P) - \sigma_{\mu\nu}\, k_\nu\, H_R(k;P)\right] \\ &\quad + \tilde{\Gamma}_{5\mu}^j(k;P) + \frac{f_{\pi_n} P_\mu}{P^2+m_{\pi_n}^2}\Gamma_{\pi_n}^j(k;P)\,, \end{aligned} \tag{7}$$

where: F_R, G_R, H_R and $\tilde{\Gamma}_{5\mu}^i$ are regular as $P^2 \to -m_{\pi_n}^2$, $P_\mu \tilde{\Gamma}_{5\mu}^i(k;P) \sim \mathrm{O}(P^2)$ and nonsingular; $\Gamma_{\pi_n}^j(k;P)$ is the 0^{-+} bound state's Bethe-Salpeter amplitude:

$$\begin{aligned} \Gamma_{\pi_n}^j(k;P) &= \tau^j\gamma_5 \left[\mathrm{i}E_{\pi_n}(k;P) + \gamma\cdot P F_{\pi_n}(k;P)\right. \\ &\quad \left. + \gamma\cdot k\, k\cdot P\, G_{\pi_n}(k;P) + \sigma_{\mu\nu}\, k_\mu P_\nu\, H_{\pi_n}(k;P)\right], \end{aligned} \tag{8}$$

which is determined by the homogeneous Bethe-Salpeter equation

$$\left[\Gamma_{\pi_n}^j(k;P)\right]_{tu} = \int_q^\Lambda [\chi_{\pi_n}^j(q;P)]_{sr}\, K_{tu}^{rs}(q,k;P)\,; \tag{9}$$

and f_{π_n} is the pseudoscalar meson's leptonic decay constant

$$f_{\pi_n}\,\delta^{ij}\,P_\mu = Z_2\,\mathrm{tr}\int_q^\Lambda \tfrac{1}{2}\tau^i\gamma_5\gamma_\mu\chi_{\pi_n}^j(q;P)\,, \tag{10}$$

where the trace is over colour, flavour and spinor indices. Equation (10) is the expression in quantum field theory for the *pseudovector* projection of the meson's Bethe-Salpeter wave function onto the origin in configuration space.

The analogous expression in the case of the pseudoscalar vertex is

$$\mathrm{i}\Gamma_5^j(k;P)\Big|_{P^2+m_{\pi_n}^2\approx 0} = \text{regular terms} + \frac{\rho_{\pi_n}}{P^2+m_{\pi_n}^2}\,\Gamma_{\pi_n}^j(k;P)\,, \tag{11}$$

$$\mathrm{i}\rho_{\pi_n}(\zeta)\,\delta^{ij} = Z_4\,\mathrm{tr}\int_q^\Lambda \tfrac{1}{2}\tau^i\gamma_5\chi_{\pi_n}^j(q;P)\,. \tag{12}$$

Equation (12) expresses the *pseudoscalar* projection of the meson's Bethe-Salpeter wave function onto the origin in configuration space.

*The hadron spectrum exhibits a sequence of $J^{PC} = 0^{-+}$ mesons, with $\pi(140)$ being the lowest mass entry. In quantum mechanical models the other members of this sequence: $\pi(1300)$, $\pi(1800)$, . . . , are described as radial excitations of the $\pi(140)$. Aspects of this interpretation persist in Poincaré covariant studies in quantum field theory and hence we retain the nomenclature.

Inserting Eqs. (7) and (11) into Eq. (5), and subsequently equating pole terms, one obtains the model-independent result [11]

$$f_{\pi_n} m_{\pi_n}^2 = [m_u(\zeta) + m_d(\zeta)]\, \rho_{\pi_n}(\zeta)\,. \tag{13}$$

In the chiral limit, Eq. (3), the axial-vector Ward-Takahashi identity becomes

$$P_\mu \Gamma^j_{5\mu}(k;P) = \mathcal{S}^{-1}(k_+)\mathrm{i}\gamma_5 \frac{\tau^j}{2} + \mathrm{i}\gamma_5 \frac{\tau^j}{2}\, \mathcal{S}^{-1}(k_-). \tag{14}$$

Assume that chiral symmetry is dynamically broken so that the dressed-quark propagator has a nonzero Dirac-scalar term, i.e., $B \neq 0$ in Eq. (2). It then follows [11] that there is a massless pseudoscalar bound state, $m_{\pi_0} = 0$, for which

$$f^0_{\pi_0} E_{\pi_0}(k;0) = B(k^2)\,, \tag{15}$$

with similar relations for the other scalar functions in Eq. (8). This is clearly the ground state, and finiteness of the right hand side in Eq. (15) entails $f^0_{\pi_0} \neq 0$. Moreover, for the ground state pion in the chiral limit [11]

$$\rho_{\pi_0}(\zeta) \to \rho^0_{\pi_0}(\zeta) = \frac{1}{f^0_{\pi_0}}\, Z_4\, \mathrm{tr} \int_q^\Lambda S^0(q) = -\frac{1}{f^0_{\pi_0}}\, \langle \bar{q}q \rangle^0_\zeta\,, \tag{16}$$

where $\langle \bar{q}q \rangle^0$ is the vacuum quark condensate. Hence the Gell-Mann–Oakes–Renner relation for the ground state pion appears as a corollary of Eq. (13). Another important corollary of Eq. (13), valid for pseudoscalar mesons containing at least one heavy-quark, is described in Ref. [12].

It is plain that $m_{\pi_{n\neq 0}} > m_{\pi_{n=0}}$ for all radial excitations and hence in the chiral limit $m_{\pi_{n\neq 0}} > 0$. In this limit it is impossible to avoid the fact that the absence of a pole contribution on the right hand side of Eq. (14) forces

$$f^0_{\pi_{n\neq 0}} = 0\,; \tag{17}$$

thus, in the chiral limit, the leptonic decay constant vanishes for every one of the pion's radial excitations.† In general

$$\frac{f_{\pi_n}}{f_{\pi_0}} = \frac{m^2_{\pi_0}}{m^2_{\pi_n}}\, \frac{\rho_{\pi_n}}{\rho_{\pi_0}}\,. \tag{18}$$

†A discussion of this result in chiral quark models is presented in Ref. [13]. We thank M.K. Volkov and V.L. Yudichev for bringing this to our attention.

3. Quantitative illustration

The manner in which these results are realised in QCD can be illustrated using the one-parameter renormalisation-group-improved ladder model for the quark-antiquark scattering kernel introduced in Ref. [14], and reviewed in Ref. [3]. One first solves the gap equation, Eq. (1), whose solution is required to complete the specification of the Bethe-Salpeter equation, Eq. (9), and then solves this for the pion and its first radial excitation.

The homogeneous Bethe-Salpeter equation is an eigenvalue problem, with the bound state masses: $P^2 = -m_{\pi_n}^2$, being the eigenvalues. We want only the first two eigenvalues and eigenvectors. They can be obtained from the modified equation

$$l(P^2)\ \left[\Gamma^j_{\pi_n}(k;P)\right]_{tu} = \int_q^\Lambda [\chi^j_{\pi_n}(q;P)]_{sr}\, K^{rs}_{tu}(q,k;P)\,, \tag{19}$$

with $l(P^2)$ a scalar, which has a solution for every value of P^2 and can therefore be solved by iteration. To explain this, consider the equation written in the form

$$l(P^2)\,|g\rangle = M(P^2)\,|g\rangle\,, \tag{20}$$

where the matrix $M(P^2)$ denotes the full kernel of the Bethe-Salpeter equation. One fixes a value of P^2 and "guesses" a solution: $|g(0)\rangle$. The kernel, $M(P^2)$, has a complete set of real eigenvectors $|g_i\rangle$ with eigenvalues λ_i, ordered such that $\lambda_0 > \lambda_1 > \ldots$, and therefore

$$|g(0)\rangle = \sum_{i=0}^{\infty} a_i |g_i\rangle\,, \tag{21}$$

where a_i are real constants and the vector is canonically normalised. It is clear that

$$M(P^2)^N\,|g(0)\rangle = \sum_{i=0}^{\infty} \lambda_i^N\, a_i|g_i\rangle \ = \lambda_0^N \left\{ a_0|g_0\rangle \ + \sum_{i=1}^{\infty} \frac{\lambda_i^N}{\lambda_0^N}\, a_i|g_i\rangle \right\} \tag{22}$$

and hence for sufficiently large N,

$$M(P^2)^{N+1}\,|g(0)\rangle \approx \lambda_0\, M(P^2)^N\,|g(0)\rangle. \tag{23}$$

Thus repeated operation of the kernel on the initial "guess" produces the largest eigenvalue, $l_0(P^2) = \lambda_0$, and its associated eigenvector to any required accuracy.

One completes this exercise for a range of values of P^2 and thereby obtains a trajectory $l_0(P^2)$ that maps the P^2-evolution of the integral equation's largest eigenvalue. It is then straightforward to find that P^2 for which $l_0(P^2) = 1$. This is the solution of Eq. (9) so that $P^2 = -m_{\pi_0}^2$ and the associated eigenvector is the ground state pion's Bethe-Salpeter amplitude.

The procedure can also be applied to determine the first excited state. One fixes $P^2 < -m_{\pi_0}^2$, and finds the largest eigenvalue and associated eigenvector for this new mass-scale. That completed, one again "guesses" the Bethe-Salpeter amplitude but now projects out the eigenvector associated with the largest eigenvalue at this P^2:

$$|\tilde{g}(0)\rangle = |g(0)\rangle - |g_0\rangle \frac{1}{\langle g_0|g_0\rangle} \langle g_0|g(0)\rangle \,. \tag{24}$$

The iterative procedure then applied as before to $|\tilde{g}(0)\rangle$ yields the second largest eigenvalue and its associated eigenvector. One thereby obtains the P^2-evolution of the second largest eigenvalue, $l_1(P^2)$. The solution of $l_1(P^2) = 1$ gives the mass of the first excited state and at this P^2 the eigenvector is that state's Bethe-Salpeter amplitude. Any finite number of excited states can be studied in this way.

We have not yet specified the meaning of the inner product used implicitly in Eq. (24). For models in the class characterised by the rainbow-ladder truncation

$$\langle h|g\rangle := \mathrm{tr} \int_q^\Lambda h(q;-P)\, S(q_+)\, g(q;P)\, S(q_-)\,. \tag{25}$$

The condition $\langle h|g\rangle = 0$ then expresses a statement that the momentum-dependent $g \to h$ vacuum polarisation (overlap amplitude) vanishes at P^2, which is akin to saying that the P^2-dependent mass-mixing matrix for the states g, h does not possess off-diagonal terms.

We have obtained the mass and amplitude for the ground state pion, using the complete expression in Eq. (8), and found: $f_{\pi_0} = 0.092\,\mathrm{GeV}$; $m_{\pi_0} = 0.14\,\mathrm{GeV}$; $\rho_{\pi_0} = (0.81\,\mathrm{GeV})^2$, at a current-quark mass $m_d(1\,\mathrm{GeV}) = m_u(1\,\mathrm{GeV}) = 5.5\,\mathrm{MeV}$, reproducing the results in Ref. [14].

Our study of the first excited state is in its early stages and hitherto we have only employed the leading amplitude, E_{π_1}, in Eq. (8), and obtained an estimate for the mass

$$m_{\pi_{n=1}} \approx 1.1\,\mathrm{GeV}. \tag{26}$$

Assuming $\rho_{\pi_{n=1}} \leq \rho_{\pi_0}$, which is supported by our preliminary estimates, it follows from Eq. (18) that at the physical current-quark mass

$$f_{\pi_{n=1}} \leq 0.016\, f_{\pi_{n=0}} = 1.5\,\mathrm{MeV}. \tag{27}$$

4. *Epilogue*

We have necessarily been brief. There are many other applications of interest to this community, among them the *ab initio* calculation of electromagnetic and transition pion form factors, and a calculation of the pion's valence-quark distribution function whose discrepancy with extant data raises difficult questions. These and other studies are reviewed in Ref. [3]. A pressing contemporary challenge is the extension of the framework to the calculation of baryon observables, aspects of which we are beginning to be understand [15].

Acknowledgements

CDR thanks the organisers for their assistance and, in particular, Dubravko Klabučar for his kindness and hospitality. We acknowledge useful conversations with D. B. Blaschke, P. C. Tandy, M. K. Volkov and V. L. Yudichev. This work was supported by: the Austrian Research Foundation FWF, Erwin-Schrödinger-Stipendium no. J2233-N08; the Department of Energy, Nuclear Physics Division, contract no. W-31-109-ENG-38; and the A. v. Humboldt-Stiftung via a F. W. Bessel Forschungspreis.

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DYSON-SCHWINGEROVE JEDNADŽBE, PIONI I POVEZANE ČINJENICE

Dajemo sažet pregled izglednosti proučavanja Dyson-Schwingerove jednadžbe (DSE) koja se odnose na pseudoskalarne mezone: potvrdu računa na rešetki da su u infracrvenom području propagatori neperturbativno obučeni, i egzaktni ishodi, npr., da u kiralnoj granici leptonska konstanta raspada trne za sve pseudoskalarne mezone izuzev piona.