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VACUUMLESS GLOBAL MONOPOLE IN EINSTEIN–CARTAN THEORY

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We analyze the gravitational field of vacuumless global monopole in the context of Einstein–Cartan theory under the weak field assumption of the field equations. It has been shown that global monopole exerts attractive gravitational force on a test particle. This effect is absent in general relativity.

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1. Introduction

At the early stages of its evolution, the Universe underwent phase transitions as a result of several topological defects, namely domain walls, cosmic strings, monopoles and textures have occured [1]. Their nature depends on the symmetry breaking in the field theory under consideration. Cosmologists have been attracted to topological defects as a possible source for density perturbations, which seeded galaxy formation [2].

A typical symmetry-breaking model is described by the Lagrangian

$$L = \frac{1}{2}\partial_\mu \Phi^a \partial^\gamma \Phi^a - V(f)\,, \tag{1}$$

where Φ^a is a set of scalar fields, $a = 1, 2, \ldots, N$, $f = \sqrt{\Phi^a \Phi^a}$ and $V(f)$ has a minimum at a non-zero value of f. The model has $O(N)$ symmetry and admits domain wall, string and monopole solutions for $N = 1, 2$ and 3, respectively.

It has recently been suggested by Cho and Vilenkin (CV) [3,4] that topological defects can also be formed in the models where $V(f)$ has a maximum at $f = 0$ and decreases monotonically to zero for $f \to \infty$ without having any minima. For example,

$$V(f) = \lambda M^{4+n}(M^n + f^n)^{-1}, \tag{2}$$

where M, λ and n are positive constants.

This type of potential can arise in non-perturbative superstring models. Defects arising in these models are termed as vacuumless.

CV have studied the gravitational field of topological defects in the above models within the framework of general relativity [3]. But at sufficiently high-energy scales, it seems likely that gravity is not given by Einstein's action. Over the last two decades, the developments in cosmology and particle physics (also in condensed matter physics) were marked by a close interaction between two fields. We know that in general relativity, matter is represented by the energy momentum tensor, which essentially provides a description of mass-density distribution in space-time. In microscopic world, one can find that matter is made of elementary particles, which follow the laws of special relativity and quantum mechanics, and each particle is characterized not only by a mass, but also by a spin (intrinsic angular momentum). So at a microscopic level, the energy-momentum tensor alone is no longer sufficient to characterize dynamically the matter sources, but also the spin density tensor is needed. Inclusion of microphysics into general theory of relativity is called Einstein–Cartan theory.

In this paper, we study the gravitational field of a vacuumless global monopole in Einstein–Cartan theory under weak field approximation of the field equations.

2. The basic equations

A global monopole is described by a triplet of scalar fields Φ^a, $a = 1, 2, 3$. The monopole ansatz is

$$\Phi^a = f(r)(x^a/r), \tag{3}$$

where r is the distance from the monopole center. For the power law potential (2), it can be verified that the field equation for $f(r)$ admits a solution of the form [3,4]

$$f(r) = aM(r/\delta)^{2/(n+2)}, \tag{4}$$

where $\delta = \lambda^{-1/2}M^{-1}$ is the core radius of the monopole; r is the distance from the monopole centre and $a = (n+2)^{2/(n+2)}(n+4)^{-1/(n+2)}$. The solution (4) applies for

$$d \ll r \ll R, \tag{5}$$

where the cut-off radius R is set by the distance to the nearest anti-monopole.

For a vacuumless monopole, the space-time is static and spherically symmetric. One can write the corresponding line element as

$$\mathrm{d}s^2 = e^{\gamma(r)}\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega_2^2. \tag{6}$$

The general energy-momentum tensor for the vacuumless monopole is given by [3,4]

$$
\begin{aligned}
T_t^t &= \frac{1}{2}\big[(f')^2/e^{\lambda(r)}\big] + \big(f^2w^2/r^2\big) + \big(1/2e^2r^2\big)\big[[(w')^2/e^{\lambda(r)}] \\
&\quad +[(1-w^2)^2/2r^2]\big] + V(f), \qquad (7)\\
T_r^r &= -\frac{1}{2}\big[(f')^2/e^{\lambda(r)}\big] + \big(f^2w^2/r^2\big) + \big(1/2e^2r^2\big)\big[[(w')^2/e^{\lambda(r)}] \\
&\quad +[(1-w^2)^2/2r^2]\big] + V(f), \qquad (8)\\
T_\theta^\theta = T_\phi{}^\phi &= \frac{1}{2}\big[(f')^2/e^{\lambda(r)}\big] + \big(1/2e^2r^2\big)\big[[(w')^2/e^{\lambda(r)}] \\
&\quad +[(1-w^2)^2/2r^2]\big] + V(f) \qquad (9)
\end{aligned}
$$

($'$ denotes the differentiation with respected to r). For monopoles, the gauge field is $A^a{}_i(r) = [1-w(r)]\varepsilon^{aij}x^j/er^2$. The expressinons (7) – (9) with $w = 1$ are for the global monopole. For global vacuumless monopole, one can use the flat space approximation for $f(r)$ in Eq. (4) for $r \gg \delta$ and the form of $V(f)$ given in Eq. (2).

Following Prasanna [5], the Einstein–Cartan equations can be written as

$$
\begin{aligned}
R_a^b - \frac{1}{2}R\delta_a^b &= -8\pi G T_a^b, \qquad (10)\\
Q_{bc}^a - \delta_b^a Q_{lc}^l - \delta_c^a Q_{bl}^l &= -8\pi G S_{bc}^a\,. \qquad (11)
\end{aligned}
$$

We assume that the spins of the particles composing the monopoles are all aligned in the r-direction.

So, the only non-zero component of the spin tensor s_{ab} is $s_{\theta\phi} = K$, say, a constant. The non-zero components of s_{bc}^a are

$$
s_{\theta\phi}^t = -s_{\phi\theta}^t = K. \qquad (12)
$$

Here s_{bc}^a is the spin density described through the relations

$$
s_{bc}^a = U^a s_{bc}, \quad \text{with} \quad U^c s_{bc} = 0, \qquad (13)
$$

where U^a is the four-velocity vector $U^a = \delta_t{}^a$. Hence from Eq.(11), we get for $Q^a{}_{bc}$ the components

$$
Q_{\theta\phi}^t = -Q_{\phi\theta}^t = -8\pi G K.
$$

The field equations for the metric (6) in Einstein–Cartan theory are

$$
\begin{aligned}
e^{-\lambda}\big[(1/r^2) - (\lambda'/r)\big] - (1/r^2) - 16\pi^2G^2K^2 &= 8\pi G C r^{-b}, \qquad (14)\\
e^{-\lambda}\big[(1/r^2) + (\gamma'/r)\big] - (1/r^2) + 16\pi^2G^2K^2 &= 8\pi G D r^{-b}, \qquad (15)\\
\frac{1}{2}e^{-\lambda}\big[\gamma'' + \frac{1}{2}(\gamma')^2 - \frac{1}{2}\gamma'\lambda' + [(\gamma'-\lambda')/r]\big] + 16\pi^2G^2K^2 &= 8\pi G E r^{-b}, \qquad (16)
\end{aligned}
$$

where

$$\begin{aligned} C &= \big[[2a^2M^2/(n+2)^2] + a^2M^2 + (M^2/a^2)\big]\delta^{-4/(n+2)}, \\ D &= \big[-[2a^2M^2/(n+2)^2] + a^2M^2 + (M^2/a^2)\big]\delta^{-4/(n+2)}, \\ E &= \big[[2a^2M^2/(n+2)^2] + (M^2/a^2)\big]\delta^{-4/(n+2)}, \quad \text{and} \quad b = 2n/(n+2). \end{aligned}$$

3. *Solutions in the weak-field approximations*

Under the weak-field approximation, one can write

$$e^{\gamma} = 1 + f, \qquad e^{\lambda} = 1 + g, \tag{17}$$

where $f, g \ll 1$.

In this approximation, Eqs. (14) – (15) take the following forms

$$(g/r^2) + (g'/r) + 16\pi^2 G^2 K^2 = -8\pi G C r^{-b}, \tag{18}$$

$$(f'/r) - (g/r^2) + 16\pi^2 G^2 K^2 = 8\pi G D r^{-b}. \tag{19}$$

Solving these equations, we get

$$g = -(16/3)\pi^2 G^2 K^2 r^2 - \big[8\pi G C/(3-b)\big] r^{2-b}, \tag{20}$$

$$f = -(32/3)\pi^2 G^2 K^2 r^2 - \big[8\pi G/(2-b)\big]\big[-D + [C/(3-b)]\big] r^{2-b}. \tag{21}$$

Thus the solution for vacuumless global monopole is

$$\begin{aligned} \mathrm{d}s^2 = {} & \big[1 - (32/3)\pi^2 G^2 K^2 r^2 - \pi G M^2 P(n)(r/\delta)^{2-b}\big]\mathrm{d}t^2 \\ & -\big[1 - (16/3)\pi^2 G^2 K^2 r^2 - \pi G M^2 F(n)(r/\delta)^{2-b}\big]\mathrm{d}r^2 - r^2\mathrm{d}\Omega_2^2, \end{aligned} \tag{22}$$

where $P(n)$ and $F(n)$ are complicated functions of n.

In the absence of the spin tensor, our metric transforms to

$$\begin{aligned} \mathrm{d}s^2 = {} & \big[1 - \pi G M^2 K(n)(r/\delta)^{2-b}\big]\mathrm{d}t^2 \\ & -\big[1 - \pi G M^2 C(n)(r/\delta)^{2-b}\big]\mathrm{d}r^2 - r^2\mathrm{d}\Omega_2^2, \end{aligned} \tag{23}$$

which is of the same form as CV 's solution in general relativity [3].

4. Gravitational effects on test particles

Let us now consider a relativistic particle of mass m, moving in the gravitational field of a monopole described by Eq. (22), using the formalism of Hamilton and Jacobi (HJ).

Accordingly, the HJ equation is [6]

$$[B(r)]^{-1}[\delta S/\delta t]^2 - [A(r)]^{-1}[\delta S/\delta r]^2 - [r^{-2}][\delta S/\delta q]^2 - [r^{-2}\sin^{-2}\theta][\delta S/\delta\phi]^2 + m^2 = 0, \quad (24)$$

where

$$B(r) = \left[1 - (32/3)\pi^2 G^2 K^2 r^2 - \pi G M^2 P(n)(r/\delta)^{2-b}\right]$$

and

$$A(r) = \left[1 - (16/3)\pi^2 G^2 K^2 r^2 - \pi G M^2 F(n)(r/\delta)^{2-b}\right].$$

In order to solve the particle differential equation, let us use the separation of variables for the HJ function S as follows [6]

$$S(t,r,\theta,\phi) = -Et + S_1(r) + S_2(\theta) + J\phi. \quad (25)$$

Here the constants E and J are identified as the energy and angular momentum of the particle. As a result, the expressions for $S_1(r)$ and $S_2(\theta)$ are

$$\begin{aligned} S_1(r) &= \epsilon \int \sqrt{A[m^2 - (p^2/r^2) + (E^2/B)]}\,\mathrm{d}r, \\ S_2(\theta) &= \epsilon \int \sqrt{p^2 - J^2 \mathrm{cosec}^2\theta}\,\mathrm{d}\theta, \end{aligned} \quad (26)$$

where $\epsilon = \pm 1$, and p is the separation constant.

To determine the trajectory of the particle following the HJ method, we consider [6]

$$(\delta S/\delta E) = const., \quad (\delta S/\delta p) = const., \quad (\delta S/\delta J) = const.$$

(we have taken the constants to be zero without any loss of generality). Hence the equations for the trajectory can be obtained as

$$t = \epsilon \int (\sqrt{A}E/B)\left[m^2 - (p^2/r^2) + (E^2/B)\right]^{-1/2}\mathrm{d}r, \quad (27)$$

$$\phi = \epsilon \int J\mathrm{cosec}^2\theta\left[p^2 - J^2\mathrm{cosec}^2\theta\right]^{-1/2}\mathrm{d}\theta, \quad (28)$$

and

$$\cos^{-1}\left(\cos\theta/\sqrt{1-(J^2/p^2)}\right) = \epsilon \int (\sqrt{A}p/r^2)\left[m^2 - (p^2/r^2) + (E^2/B)\right]^{-1/2}\mathrm{d}r. \quad (29)$$

Hence the radial velocity of the particle is

$$(\mathrm{d}r/\mathrm{d}t) = (B/E\sqrt{A})\sqrt{E^2B^{-1} + m^2 - (p^2/r^2)}. \tag{30}$$

The turning points of the trajectory are given by $(\mathrm{d}r/\mathrm{d}t) = 0$ and as a consequence the potential curves are

$$(E/m) = \sqrt{B\left[(p^2/m^2r^2) - 1\right]}. \tag{31}$$

Thus

$$\begin{aligned}(E/m) &= \sqrt{1 - (32/3)\pi^2G^2K^2r^2 - \pi GM^2P(n)(r/\delta)^{2-b}} \\ &\quad \times\sqrt{(p^2/m^2r^2) - 1}.\end{aligned} \tag{32}$$

In this case the extrema of the potential curve are solutions of the equation

$$m^2H(2-b)r^{4-b} + bHr^{2-b} + (64/3)m^2\pi^2G^2K^2r^4 - 2p^2 = 0, \tag{33}$$

where $H = \pi GM^2P(n)\delta^{b-2}$.

This equation has at least one positive real root provided $2 - b$ is an positive integer. So it is possible to have a bound orbit of the test particle. Thus the gravitational field of the global monopole is shown to be attractive in nature, but here we have to impose some restriction on the constant n. The vacuumless global monopole in general relativity exerts no gravitational force [4].

5. *Conclusions*

This work extends the earlier work by Cho and Vilenkin regarding the gravitational field of a vacuumless monopole to Einstein–Cartan theory. We see that in going from general relativity to Einstein–Cartan theory, both space-time curvature and topology are affected by the presence of the spin tensor. Our study of the motion of the test particle reveals that the vacuumless global monopole in Einstein–Cartan theory exerts gravitational force, which is attractive in nature. It is dissimilar to the case of a vacuumless global monopole in general relativity. So this observation is in striking contrast with the analogue in Einstein's theory.

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BEZVAKUUMSKI GLOBALNI MONOPOL U EINSTEIN–CARTANOVOJ TEORIJI

Analiziramo gravitacijsko polje bezvakuumskog globalnog monopola u okviru Einstein–Cartanove teorije, uzevši pretpostavku slabog polja u jednadžbama polja. Pokazuje se da globalni monopol proizvodi privlačnu gravitacijsku silu na ispitnu česticu. Takvog učinka nema u općoj teoriji relativnosti.