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A STUDY ON BIANCHI-IX COSMOLOGICAL MODEL IN LYRA GEOMETRY

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Some cosmological phenomena are studied from modified Einstein's equations based on Lyra geometry in Bianchi-IX space-time. We study the model in the presence of a massless scalar field with a flat potential.

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1. Introduction

The origin of structure in the Universe is one of the greatest cosmological mysteries. The exact physical situation at very early stages of the formation of our universe is still unknown. The present day observations, based on astronomical observetions, indicate that large-scale structure of the Universe is homogeneous and isotropic. But no definite evidence is known that the early Universe had the same properties. Perhaps in the early era, some other type of cosmological model evolved and then at some stage changed over to the present-day homogeneous model. Hence it is reasonably relevant to study different Bianchi-type models of the Universe. So far the isotropic cosmological models have been studied extensively. There are only a few works on anisotropic models, essentially on Bianchi-IX cosmological model, due to the complicated nature of the field equations. Actually, the study of anistropic models were introduced after the discovery of microwave background radiation in 1965. It was found that the radiation was isotropic to one part in 10^4, apart from a dipole anisotropy which was attributed to the peculiar motion of our galaxy [1]. Since the discovery of general relativity by Einstein, there have been numerous modification of his theory. Lyra [2] suggested a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance's to Weyl's geometry.

In subsequent investigations, Sen [3] and Sen and Dunn [4] proposed a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein field equations based on Lyra's geometry which in normal gauge may be written as

$$R_{ik} - \frac{1}{2} g_{ik} R + \frac{3}{2} {}^{*}\phi_i {}^{*}\phi_k - \frac{3}{4} g_{ik} {}^{*}\phi_j {}^{*}\phi^j \quad = \quad 8\pi G T_{ik}, \tag{1}$$

where ${}^{*}\phi_i$ is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

According to Halford [5], this theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of the field equations.

Further investigations were done by several authors in scalar-tensor theory and cosmology within the framework of Lyra geometry [6].

However, Soleng [7] has pointed out that the cosmologies based on Lyra's manifold with constant gauge vector ${}^{*}\phi_i$ will either include a creation field and be equal to Hoyle's creation field cosmology, or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

In this paper, we study the cosmological model for Bianchi IX space-time within the framework of Lyra geometry in the presence of a massless scalar field with a flat potential.

2. *Field equations*

The time-like displacement vector ${}^{*}\phi_i$ in Eq. (1) is given by

$${}^{*}\phi_i \quad = \quad \big(\beta(t), 0, 0, 0\big). \tag{2}$$

The Lagrangian will be that of gravity minimally coupled to a scalar field $\phi(t)$ with potential $V(\phi)$ [8]

$$L \quad = \quad \int \sqrt{-g} \left[R - \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - V(\phi) \right] \mathrm{d}^4 x, \tag{3}$$

where $g = \det(g_{ab})$ and R is the Ricci scalar. Hence the energy-momentum tensor has the form [8]

$$T_{ab} \quad = \quad \frac{1}{2} \partial_a \phi \partial_b \phi - \left[\frac{1}{4} \partial_k \phi \partial^k \phi + \frac{1}{2} V(\phi) \right] g_{ab}. \tag{4}$$

The ϕ-field equation is

$$\frac{1}{\sqrt{-g}} \partial_a \big(\sqrt{-g}\, (\partial^a \phi) \big) \quad = \quad - \frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{5}$$

The Bianchi-type IX metric is taken as [9]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2\mathrm{d}x^2 + b^2\mathrm{d}y^2 + \big(b^2\sin^2 y + a^2\cos^2 y\big)\mathrm{d}z^2 - 2a^2\cos y\,\mathrm{d}x\mathrm{d}z, \tag{6}$$

where a, b are functions of time coordinate only.

The field equations (1) for the metric (6) are (ϕ will be rescaled by 2ϕ)

$$2\frac{a'b'}{ab} + \frac{(b')^2}{b^2} + \frac{1}{b^2} - \frac{1}{4}\frac{a^2}{b^4} - \frac{3}{4}\beta^2 = \big(\phi'\big)^2 + \frac{1}{2}V(\phi), \tag{7}$$

$$2\frac{b''}{b} + \frac{(b')^2}{b^2} + \frac{1}{b^2} - \frac{3}{4}\frac{a^2}{b^4} + \frac{3}{4}\beta^2 = -\big(\phi'\big)^2 + \frac{1}{2}V(\phi), \tag{8}$$

$$\frac{a''}{a} + \frac{b''}{b} + \frac{a'b'}{ab} + \frac{1}{4}\frac{a^2}{b^4} + \frac{3}{4}\beta^2 = -\big(\phi'\big)^2 + \frac{1}{2}V(\phi), \tag{9}$$

$$\phi'' + \Big(\frac{a'}{a} + 2\frac{b'}{b}\Big)\phi' = -\frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{10}$$

(here $'$ denotes differentiation with respect to t, and we have chosen $8\pi G = 1$). The expansion scalar θ and shear σ^2 are given by

$$\theta = \frac{a'}{a} + 2\frac{b'}{b}\,, \qquad \sigma^2 = \frac{2}{3}\Big(\frac{a'}{a} - \frac{b'}{b}\Big)^2. \tag{11}$$

3. Solutions to the field equations

The potential can be approximated by a constant value (cf. Schabes [8])

$$V(\phi) = 2\lambda. \tag{12}$$

It may be noted that the coefficient of ϕ' in Eq. (10) acts as a friction term and it is larger for an isotropic model. So the ϕ-field moves slowly in an anisotropic space-time. Now Eq. (10) can easily be integrated to give

$$\phi' = \frac{\phi_0}{ab^2}, \tag{13}$$

where ϕ_0 is an integration constant. From the field equations (8) and (9), we get

$$\frac{b''}{b} - \frac{a''}{a} + \frac{(b')^2}{b^2} - \frac{a'b'}{ab} - \frac{a^2}{b^4} + \frac{1}{b^2} = 0. \tag{14}$$

To solve the differential equation, one may assume the following relation between the metric coefficients

$$a = b^n, \tag{15}$$

where n is an arbitrary constant. So by use of (15), one obtains from Eq. (14) the differential equation for b as

$$\frac{b''}{b} + (n+1)\frac{(b')^2}{b^2} = \frac{1}{(n-1)b^2} - \frac{1}{n-1}b^{2n-4}. \tag{16}$$

The equation has a first integral

$$(b')^2 = \frac{1}{n^2-1} - \frac{1}{2n^2-2n}b^{2n-2} + Db^{-2n-2}, \tag{17}$$

where D is an integration constant. This first-order equation can be written in an integral form as

$$\int \left[\frac{1}{n^2-1} - \frac{1}{2n^2-2n}b^{2n-2} + Db^{-2n-2}\right]^{-1/2} \mathrm{d}b = \pm\big(t-t_0\big) \tag{18}$$

(t_0 is an integration constant). This equation can't be solved for arbitrary values of n and D. We have found solutions only for $D = 0$ and $n = 2$, $1/2$, $3/2$ and $3/4$.

3.1. Case I, $n = 2$

In this case, the expression for b from the integral (18) will be

$$b = \frac{2}{\sqrt{3}}\sin\frac{t-t_0}{2}. \tag{19}$$

The other physically important quantities are obtained as

$$a = \frac{4}{3}\sin^2\frac{t-t_0}{2}, \tag{20}$$

$$\theta = 2\cot\frac{t-t_0}{2}, \tag{21}$$

$$\sigma^2 = \frac{1}{6}\cot^2\frac{t-t_0}{2}, \tag{22}$$

$$\phi = \phi_{00} - \phi_0\big[\frac{2}{3}\mathrm{cosec}^2\frac{t-t_0}{2}\cot\frac{t-t_0}{2} + \frac{4}{3}\cot\frac{t-t_0}{2}\big], \tag{23}$$

$$\frac{3}{4}\beta^2 = \frac{1}{4}\mathrm{cosec}^2\frac{t-t_0}{2}\big[5\cos^2\frac{t-t_0}{2} + 3\big] - \frac{81\phi_0^2}{256}\mathrm{cosec}^8\frac{t-t_0}{2} - \frac{1}{4} - \lambda \tag{24}$$

(ϕ_{00} is an integration constant).

3.2. *Case II, $n = 1/2$*

From Eq. (18), we obtain the expression for b as

$$-\sqrt{b(\frac{3}{2}-b)}+\frac{3}{4}\sin^{-1}\sqrt{\frac{2b}{3}} \quad = \quad \frac{1}{\sqrt{3}}(t-t_0). \tag{25}$$

But in this case, we can't get explicit form of b in terms of t and consequently all physical parameters can't be determined in terms of t. So no conclusion can be drawn.

3.3. *Case III, $n = 3/2$*

The solution for b in Eq. (18) is

$$b \quad = \quad \frac{8}{15}-\frac{3}{16}(t-t_0)^2. \tag{26}$$

The solution shows a contracting model and is not of much physical interest.

3.4. *Case IV, $n = 3/4$*

Here we get the following solution for b

$$b \quad = \quad \left(S+T-\frac{3}{2}\right)^2, \tag{27}$$

where

$$S \quad = \quad \left[\frac{1}{2}C-\frac{8}{27}+\sqrt{\left(\frac{1}{2}C-\frac{8}{27}\right)^2-\frac{64}{729}}\right]^{1/3}, \tag{28}$$

$$T \quad = \quad \left[\frac{1}{2}C-\frac{8}{27}-\sqrt{\left(\frac{1}{2}C-\frac{8}{27}\right)^2-\frac{64}{729}}\right]^{1/3}, \tag{29}$$

$$C \quad = \quad \frac{96}{7}+\frac{14}{3}(t-t_0)^2. \tag{30}$$

The other parameters are obtained as before: namely,

$$a \quad = \quad \left(S+T-\frac{3}{2}\right)^{8/3}, \tag{31}$$

$$\theta \quad = \quad \frac{11}{2}\left(S+T-\frac{3}{2}\right)^{-1}\left(\frac{\mathrm{d}S}{\mathrm{d}t}+\frac{\mathrm{d}T}{\mathrm{d}t}\right), \tag{32}$$

where

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{3}\big(P+F\big)^{-2/3}\left(\frac{\mathrm{d}P}{\mathrm{d}t}+\frac{\mathrm{d}F}{\mathrm{d}t}\right), \tag{33}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{3}\big(P-F\big)^{-2/3}\left(\frac{\mathrm{d}P}{\mathrm{d}t}-\frac{\mathrm{d}F}{\mathrm{d}t}\right), \tag{34}$$

and

$$P = \frac{1}{2}C-\frac{8}{27}, \qquad F=\sqrt{\big(\frac{1}{2}C-\frac{8}{27}\big)^2-\frac{64}{729}},$$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{14}{3}(t-t_0), \qquad \frac{\mathrm{d}F}{\mathrm{d}t}=\frac{14}{3}(t-t_0)\frac{P}{F}, \tag{35}$$

$$\sigma^2 = \frac{1}{24}\theta^2, \tag{36}$$

$$\phi = \phi_{00}-\phi_0\int\big(S+T-\frac{3}{2}\big)^{-11/2}\mathrm{d}t. \tag{37}$$

Though we can't find the exact form of ϕ, one can note that it is a convergent integral,

$$\begin{aligned}\frac{3}{4}\beta^2 = {} & 14\left(\frac{\mathrm{d}S}{\mathrm{d}t}+\frac{\mathrm{d}T}{\mathrm{d}t}\right)^2\big(S+T-\frac{3}{2}\big)^{-2}+\big(S+T-\frac{3}{2}\big)^{-4} \\ & -\frac{1}{4}\big(S+T-\frac{3}{2}\big)^{-10/3}-\big(\phi_0\big)^2\big(S+T-\frac{3}{2}\big)^{-11}-\lambda. \end{aligned} \tag{38}$$

4. Concluding remarks

In this paper, we have obtained several sets of explicit solutions in Bianchi-type IX model within the framework of the Lyra geometry. For the case $n = 2$, we observe that the initial epoch will be $t = t_0$. The model starts with an initial singularity $\sqrt{-g} \to 0$, while θ and σ^2 diverge. In fact, it is a point singularity as $a, b \to 0$ at this epoch.

For the case $n = 3/4$, we note that the solution in (27) and (31) describes a nonsingular space-time. It is easy to verify that all physical quantities in (32) and (36)–(38) remain finite and regular for the entire range of variable $-\infty < t < \infty$. This indicates clearly that the model is free of singularity. Thus, one may note that singularity-free solution exists in Bianchi-IX cosmological model based on Lyra geometry. In this case, the Bianchi-IX cosmological model based on Lyra geometry shows that β takes imaginary values at $t \to \infty$. So, the concept of Lyra geometry will not linger for an infinite time.

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PROUČAVANJE KOZMOLOŠKOG MODELA BIANCHI-IX U LYRA GEOMETRIJI

Neke se kozmološke pojave tumače Einsteinovim jednadžbama zasnovanim na Lyra geometriji u vremenu-prostoru Bianchi-IX. Proučavamo taj model u prisutnosti bezmasenog skalarnog polja sa stalnim potencijalom.