ISSN 1330–0016
CODEN FIZBE7

HIGHER-DIMENSIONAL GLOBAL MONOPOLE IN LYRA'S GEOMETRY

FAROOK RAHAMAN

Khodar Bazar, Baruipur - 743302, 24 Parganas (South), West Bengal, India
E-mail address: farook_rahaman@yahoo.com

Received 19 May 2002; Accepted 31 January 2003
Online 7 February 2003

We present an approximate solution around a global monopole resulting from a breaking of a global SO(3) symmetry in a five-dimensional space-time based on Lyra's geometry in normal gauge, i.e., displacement vector $\phi_i = (0, 0, 0, 0, \beta_0)$, where β_0 is a constant. Acceleration due to the monopole has been evaluated by studying the geodesic equation. A comparison is made with the classical results.

PACS numbers: 98.80.cq, 04.20.jb, 04.50 UDC 530.12

Keywords: breaking of a global SO(3) symmetry, global monopole, five-dimensional space-time, Lyra's geometry, geodesic equation

1. *Introduction*

Recent attempts to unify gravity with other fundamental forces in nature reveal that it is very interesting to study models where the space-time dimension is different from four. Also latest developments in super-string and Yang-Mills supergravity theory demand more than usual 4-dimensions of space-time. Solutions of Einstein's field equations are believed to be of physical relevance possibly at the extremely early times before the Universe underwent the compactification transitions [1].

Phase transitions in the early Universe can give rise to topological defects of various kinds. Recently, Pando, Valls-Gaboud and Fang [2] have proposed that the topological defects are responsible for the structure formation of our Universe. Monopoles are localized defects, and will arise if the manifold M contains surfaces which can not be continuously shrunk to a point, i.e., when $\pi_2(\mathrm{M})\neq 1$.

Global monopoles, predicted to exist in the grand unified theory, are important objects for particle physicists and cosmologists. Their energy (mass) is almost entirely concentrated in a small region near the monopole core [3]. In 1989, Barriola and Vilenkin (BV) [4] have shown an approximate solution of the Einstein's equations for the metric outside a global monopole, resulting from a global SO(3) symmetry breaking. Banerji et al. [5] have extended the work of BV to higher dimension.

In 1951, Lyra [6] proposed a modification Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry. Sen [7] and Sen and Dunn [7] proposed a new scalar tensor theory of gravitation and constructed an analog of the Einstein's field equation based on Lyra's geometry which in normal gauge may be written as

$$R_{ik} - \frac{1}{2} g_{ik} R + \frac{3}{2} \phi_i \phi_k - \frac{3}{4} g_{ik} \phi_m \phi^m = -8\pi T_{ik} \,, \tag{1}$$

where ϕ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [9] pointed out that the constant displacement field ϕ_i in Lyra's geometry plays the role of the cosmological constant Λ in the normal general relativistic treatment. According to Halford, the present theory predicts the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearised form of field equations. Soleng [8] has pointed out that the constant displacement field in Lyra's geometry will either include a creation field and be equal to Hoyle's creation-field cosmology or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term.

Subsequent investigations were done by several authors in scalar-tensor theory and cosmology within the frame work of Lyra's geometry [9].

Recently, I have obtained the weak-field approximate solution of the global monopole in Lyra's geometry and have shown that the nature of the monopole solution has changed due to the consideration based on Lyra's geometry [10]. In this work, we shall deal with the higher-dimensional global monopole with a constant displacement vector based on Lyra's geometry in normal gauge, i.e., the displacement vector $\phi_i = (\beta_0, 0, 0, 0, 0)$, and look forward whether the monopole shows any significant properties due to the introduction of the gauge field in the Riemannnian geometry.

2. *The basic equations*

The metric ansatz describing a monopole in a five-dimensional space-time can be written as

$$\mathrm{d}s^2 = \mathrm{e}^{\gamma} \mathrm{d}t^2 - \mathrm{e}^{\beta} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega_2^2 - \mathrm{e}^{\mu} \mathrm{d}\Psi^2 \,. \tag{2}$$

Here γ, β and μ are functions of r alone and Ψ is the fifth coordinate.

We closely follow the formalism of Banerji et al. [5] and take the Lagrangian as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi^a \partial^\gamma \Phi^a - \frac{1}{4} \lambda (\Phi^a \Phi^a - \eta^2)^2 \,, \tag{3}$$

where Φ^a is the triplet scalar field, $a = 1, 2, 3$ and η is the energy scale of symmetry breaking.

The field configuration is taken to be $\Phi^a = \eta f(r) x^a / r$, where $x^a x^a = r^2$.

The energy momentum tensors can be written [5] as

$$T_{ab} = 2\, \partial \mathcal{L} / \partial g^{ab} - \mathcal{L} g_{ab}$$

$$T_t^t = \eta^2 \frac{f'^2}{2\mathrm{e}^\beta} + \eta^2 \frac{f^2}{r^2} + \frac{1}{4}\lambda(\eta^2 f^2 - \eta^2)^2\,, \tag{4}$$

$$T_r^r = -\eta^2 \frac{f'^2}{2\mathrm{e}^\beta} + \eta^2 \frac{f^2}{r^2} + \frac{1}{4}\lambda(\eta^2 f^2 - \eta^2)^2\,, \tag{5}$$

$$T_\theta^\theta = T_\phi^\phi = \eta^2 \frac{f'^2}{2\mathrm{e}^\beta} + \frac{1}{4}\lambda(\eta^2 f^2 - \eta^2)^2\,, \tag{6}$$

$$T_\Psi^\Psi == \eta^2 \frac{f'^2}{2\mathrm{e}^\beta} + \eta^2 \frac{f^2}{r^2} + \frac{1}{4}\lambda(\eta^2 f^2 - \eta^2)^2\,. \tag{7}$$

(Prime denotes the differentiation with respect to r.)

It can be shown that in a flat space, the monopole core has the size $\delta \sim \sqrt{\lambda}\eta^{-1}$ and the mass $M_{\mathrm{core}} \sim \lambda^{-1/2}\eta$. Thus, if $\eta \ll m_{\mathrm{P}}$, where m_{P} is the Planck mass, it is evident that we can still apply the flat-space approximation for δ and M_{core}. This follows from the fact that in this case the gravity would not much influence the monopole structure.

Banerji et al. assumed that $f = 1$ outside the monopole core [5].

With this result, the energy stress tensors assume the following form

$$T_t^t = T_r^r = T_\Psi^\Psi = \eta^2 / r^2 \quad \text{and} \quad T_\theta^\theta = T_\phi^\phi = 0\,. \tag{8}$$

The field equations in the normal gauge for Lyra's geometry for the metric (2) reduce to

$$-\mathrm{e}^{-\beta}\left(\frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\mu'\beta'}{4} - \frac{\beta'}{r} + \frac{\mu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} - \frac{3}{4}\beta_0^2 \mathrm{e}^{-\gamma} = \frac{8\pi\eta^2}{r^2}\,, \tag{9}$$

$$-\mathrm{e}^{-\beta}\left(\frac{\mu'}{r} + \frac{\mu'\gamma'}{4} + \frac{\gamma'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{3}{4}\beta_0^2 \mathrm{e}^{-\gamma} = \frac{8\pi\eta^2}{r^2}\,, \tag{10}$$

$$-\mathrm{e}^{-\beta}\left(\frac{\gamma''}{2} + \frac{\gamma'^2}{4} + \frac{\mu'}{2r} - \frac{\beta'}{2r} + \frac{\gamma'}{2r} + \frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\mu'\beta'}{4} + \frac{\mu'\gamma'}{4} - \frac{\gamma'\beta'}{4}\right) + \frac{3}{4}\beta_0^2 \mathrm{e}^{-\gamma} = 0\,, \tag{11}$$

$$-\mathrm{e}^{-\beta}\left(\frac{\gamma''}{2} + \frac{\gamma'^2}{4} - \frac{\beta'}{r} + \frac{\gamma'}{r} - \frac{\gamma'\beta'}{4} + \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{3}{4}\beta_0^2 \mathrm{e}^{-\gamma} = \frac{8\pi\eta^2}{r^2}\,, \tag{12}$$

3. Solutions in the weak field approximations

At this stage, let us consider the weak-field approximation assuming that

$$e^{\gamma} = 1 + f(r), \quad e^{\beta} = 1 + g(r), \quad e^{\mu} = 1 + h(r)\,. \tag{13}$$

Here the functions f, g and h should be computed to first order in η^2 and β_0^2. In this approximation, Eqs. (9) to (12) take the following forms

$$-\frac{h''}{2} + \frac{g'}{r} - \frac{h'}{r} + \frac{g}{r^2} - \frac{3}{4}\beta_0^2 = \frac{8\pi\eta^2}{r^2}\,, \tag{14}$$

$$-\frac{f'}{r} - \frac{h'}{r} + \frac{g}{r^2} + \frac{3}{4}\beta_0^2 = \frac{8\pi\eta^2}{r^2}\,, \tag{15}$$

$$-\frac{f''}{2} - \frac{1}{2}\frac{f'}{r} + \frac{1}{2}\frac{g'}{r} - \frac{1}{2}\frac{h'}{r} - \frac{h''}{2} + \frac{3}{4}\beta_0^2 = 0\,, \tag{16}$$

$$-\frac{f''}{2} - \frac{f'}{r} + \frac{g'}{r} + \frac{g}{r^2} + \frac{3}{4}\beta_0^2 = \frac{8\pi\eta^2}{r^2}\,, \tag{17}$$

From Eqs. (14) and (21), we get

$$r^2(f' - h') = \beta_0^2 r^2 + a\,, \tag{18}$$

where a is an integration constant.

However, for the economy of space, we will skip all mathematical details and give the final results as

$$f = \beta_0^2 r^2/2 - a/(3r)\,, \tag{19}$$

$$h = 2a/(3r)\,, \tag{20}$$

$$g = \beta_0^2 r^2/4 + 8\pi\eta^2 - a/(3r)\,. \tag{21}$$

Thus, in the weak-field approximation, the higher-dimensional monopole metric in Lyra's geometry takes the following form

$$ds^2 = (1 + \frac{\beta_0^2 r^2}{2} - \frac{a}{3r})dt^2 - (1 + \frac{\beta_0^2 r^2}{4} + 8\pi\eta^2 - \frac{a}{3r})dr^2 - r^3\Omega_2^2 - (1 + \frac{2a}{3r})d\Psi^2\,. \tag{22}$$

4. *Motion of a test particle*

Let us consider a relativistic particle of mass m moving in the gravitational field of the monopole described by Eq. (26). The Hamilton-Jacobi (H-J) equation is [11]

$$\frac{1}{A}\left(\frac{\partial S}{\partial t}\right)^2-\frac{1}{B}\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2}\left[\left(\frac{\partial S}{\partial x_1}\right)^2+\left(\frac{\partial S}{\partial x_2}\right)^2\right]-\frac{1}{C}\left(\frac{\partial S}{\partial \Psi}\right)^2+m^2=0\,, \quad (23)$$

where

$$A=1+\beta_0^2r^2/2-a/(3r)\,,$$

$$B=1+\beta_0^2r^2/4+8\pi\eta^2-a/(3r)\,,$$

$$C=1+a/(3r)\,.$$

x_1 and x_2 are the coordinates on the surface of the 2-sphere.

Take the ansatz

$$S(t,r,x_1,x_2,\Psi)=-Et+S_1(r)+p_1x_1+p_2x_2+J\Psi \quad (24)$$

as the solution to the H-J Eq. (23). Here the constants E and J are identified as the energy and five-dimensional velocity and p_1, p_2 are momentum of the particle along different axes on the 2-sphere, with $p=(p_1^2+p_2^2)^{1/2}$ as the resulting momentum of the particle.

Substituting (24) in (23), we get

$$S_1(r)=\epsilon\int\sqrt{B(E^2/A-p^2/r^2-J^2/C+m^2)}\,\mathrm{d}r \quad \text{where} \quad \epsilon=\pm1\,. \quad (25)$$

In the H-J formalism, the path of the particle is characterized by [11]

$$\partial S/\partial E=constant,\quad \partial S/\partial p_i=constant\ \ (i=1,2),\quad \partial S/\partial E=constant\,. \quad (26)$$

Thus we get (taking the constants to be zero without any loss of generality),

$$t=\epsilon\int\frac{\sqrt{B}\,E/A}{\sqrt{E^2/A-p^2/r^2-J^2/C+m^2}}\,\mathrm{d}r\,, \quad (27)$$

$$x_i=\epsilon\int\frac{\sqrt{B}\,p_i/r^2}{\sqrt{E^2/A-p^2/r^2-J^2/C+m^2}}\,\mathrm{d}r\,, \quad (28)$$

$$\Psi=\epsilon\int\frac{\sqrt{B}\,J/C}{\sqrt{E^2/A-p^2/r^2-J^2/C+m^2}}\,\mathrm{d}r\,, \quad (29)$$

From (27), we get the radial velocity as

$$\mathrm{d}r/\mathrm{d}t = (A/(\sqrt{B}E))\sqrt{E^2/A - p^2/r^2 - J^2/C + m^2}\,. \tag{30}$$

The turning points of the trajectory are given by $\mathrm{d}r/\mathrm{d}t = 0$, and, as a consequence, the potential curves are

$$E/m = \sqrt{A\,(p^2/(mr^2) + J^2/(Cm^2) - 1)}\,. \tag{31}$$

Thus, the extrema of the potential curves are the solutions of the equation

$$\begin{aligned}
&54(J^2-m^2)\beta_0^2 r^7 - [36a(2m^2-J^2) + 6aJ^2m^2)]\beta_0^2 r^6 - [24a^2m^2 - 54p^2 + 18p^2]\beta_0^2 r^5 \\
&+[36ap^2\beta_0^2 + 18a(J^2-m^2) + 12ap^2\beta_0^2 + 36aJ^2m^2]r^4 - [12a^2J^2 + 12a^2m^2 \\
&-16a^2p^2\beta_0^2 + 108p^2]r^3 + [18ap^2 - 8a^3m^2 - 108ap^2]r^2 + 24p^2a^2r + 24p^2a^3 = 0\,.
\end{aligned} \tag{32}$$

This is an algebraic equation of odd degree (degree 7), whose last term is negative provided the integration constant a is negative. This equation has at least one real positive solution. So, it is possible to have bound orbit for the test particle, i.e., the particle can be trapped by the global monopole. In other words, our higher-dimensional monopole always exerts gravitational force which is attractive in nature.

5. *Concluding remarks*

In this paper, we have obtained an approximate solution around a global monopole resulting from the breaking of a global SO(3) symmetry in a five-dimensional space-time based on Lyra's geometry. We see that our higher-dimensional monopole metric is unique, whereas in Einstein's theory, the Banerji et al. monopole is not unique [5]. We have shown that the higher-dimensional monopole in Lyra's geometry always exerts a gravitational force which is attractive in nature, whereas the classical higher-dimensional monopole exerts gravitational force provided some restrictions are imposed [5].

I have shown previously that in case of four dimensions, the gravitational field of the monopole is changed due to the consideration of Lyra's geometry [10].

In the classical higher-dimensional monopole, g_{55} looses its dynamical role via a specific choice of an arbitrary constant [5]. Also in the higher-dimensional monopole based on Lyra's geometry, g_{55} looses its dynamical role for a particular choice of the arbitrary integration constant, say $a = 0$ (see Eq. (22)).

Here we also see that in $\Psi = const$ hyper-surfaces, our solution does reduces to that obtained by Farook Rahaman [10]. And at the same time, we note that in the absence of the displacement vector, our solution does not change to the Banerji et al. solution.

To conclude, in this work, we have extended the higher-dimensional monopole solution in Einstein's theory to the scalar-tensor theory based on Lyra's geometry.

References

[1] S. Weinberg, *Physics in Higher Dimension*, World Scientific, Singapore (1986).

[2] J. Pando, D. Valls-Gaboud and L. Fang, Phys. Rev. Lett. **81** (1998) 8568.

[3] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological defects*, Cambridge Univ. Press, Cambridge (1994).

[4] M. Bariolla and A. Vilenkin, Phys. Rev. Lett. **63** (1989) 341.

[5] A. Banerji, S. Chaterji and A. A. Sen, Class. Quan. Grav. **13** (1996) 3141.

[6] G. Lyra, Math. Z. **54** (1951) 52.

[7] D. K. Sen, Phys. Z. **149** (1957) 311; D. K. Sen and K. A. Dunn, J. Math. Phys. **12** (1971) 578 (For brief notes on Lyra geometry see also A. Beesham, Aust. J. Phys. **41** (1988) 833; T. Singh and G. P. Singh, Int. J. Th. Phys. **31** (1992) 1433).

[8] H. H. Soleng, Gen. Rel. Grav. **19** (1987) 1213.

[9] W. D. Halford, Aust. J. Phys. **23** (1970) 863; K. S. Bharma, Aust. J. Phys. 27 (1974) 541; T. M. Karade and S. M. Borikar, Gen Rel. Grav. **1** (1978) 431; A Beesham, Ast. Sp. Sc. **127** (1986) 189; T. Singh and G. P. Singh, J. Math. Phys. **32** (1991) 2456; G. P. Singh and K. Desikan, Pramana **49** (1997) 205. F. Rahman and J. Bera, Int. J. Mod. Phys. D **10** (2001) 729; Astr. Sp. Sc. (to appear in 2002).

[10] F. Rahaman, Int. J. Mod. Phys. D **9** (2000) 775; Ast. Sp. Sc. **280** (2002) 337.

[11] S. Chacraborty, Gen. Rel. Grav. **28** (1996) 1115.

VIŠEDIMENZIJSKI GLOBALNI MONOPOL U LYRINOJ GEOMETRIJI

Dajemo približno rješenje oko globalnog monopola koje slijedi kršenje globalne SO(3) simetrije u peto-dimenzijskom prostoru-vremenu, zasnovano na Lyrinoj geometriji u normalnoj baždarnosti, tj., pomačni je vektor $\phi_i = (0, 0, 0, 0, \beta_0)$, uz $\beta_0 = const.$ Ubrzanje zbog monopola određuje se promatranjem geodetske jednadžbe. Načinili smo usporedbu s klasičnim rezultatima.