ISSN 1330–0016
CODEN FIZBE7

CASIMIR EFFECT IN FOUR SITUATIONS – INCLUDING A NONCOMMUTATIVE TWO-SPHERE

MICHAL DEMETRIAN

Department of Theoretical Physics, Faculty of Mathematics, Physics and Informatics,
Mlynska Dolina F2, SK-842 48 Bratislava IV, Slovak Republic
E-mail address: demetrian@sophia.dtp.fmph.uniba.sk

Received 3 July 2002; Accepted 16 December 2002
Online 7 February 2003

A computation of the Casimir effect for a real scalar field in four situations: on a segment of a line, on a circle and on both standard commutative and noncommutative two-spheres is presented. The main aim of this paper is to discuss the Casimir energy on a noncommutative sphere within the theory with commutative time. A discussion of the (non)commutative cylinder is also given.

PACS numbers: 03.70.+k, 11.10.-z UDC 537.86

Keywords: Casimir effect, real scalar field, segment of a line, circle, commutative and noncommutative two-spheres, (non)commutative cylinder

1. *Introduction*

The energy of the vacuum, the zero-point energy, is a direct consequence of quantum mechanics. Since the birth of quantum mechanics, a central question has been whether this vacuum-state energy could have physical (measurable) consequences. Casimir effect is one of the well-known dynamical effects of the quantum vacuum state. Its importance follows from the fact that this effect was not only theoretically predicted but it has also been experimentally verified. The first time the attraction of two parallel neutral conducting plates due to electromagnetic vacuum fluctuations was predicted and then a large set of different experiments about the Casimir's idea [1] was done by many cooperating experimentalists and theoreticians (for more information see, e.g., Refs. [2, 6]). The vacuum-state energy should be of interest from the point of view of modern cosmology due to the production of a nonzero cosmological constant [7], which can drive the inflation process.

In this paper, we are interested in the vacuum-state energy of a real scalar field Φ in four situations which differ in a way in which the nonzero vacuum energy is caused. In Sect. 2, the scalar field on an interval (finite segment of a line) is discussed. The nonzero vacuum-state energy appears as the consequence of the boundaries. This example is most similar to the original Casimir's idea [1] confirmed recently [16] by the measurement of attraction of uncharged conducting plates. In Sects. 3 and 4, the Casimir's energy of the scalar field on a circle (one dimensional sphere) and a two dimensional sphere is discussed. In these cases, the nontrivial properties of vacuum state follow from the nontrivial (different from the Minkowski space) topology. In Sect. 5, the new ideas of noncommutative geometry are implemented: Casimir's energy is computed on a two dimensional fuzzy-sphere. In this case, "the problems" with renormalization and regularization disappear. Noncommutativity yields a natural cut-off by introducing a new fundamental constant, fundamental in the sense of being similar to the Planck constant. In the last section, we discuss some specific questions concerning the Casimir effect on the noncommutative sphere and also on the noncommutative cylinder.

2. *Scalar field on an interval*

We start with a real scalar field $\Phi = \Phi(t, x)$ defined on a space interval of length a: $x \in \langle 0, a\rangle$. It means that our space-time is a strip of two dimensional Minkowski space[1] with a space size equal to a. The dynamics of the classical field Φ with the mass m is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\frac{m^2c^2}{\hbar^2}\Phi^2 \,, \tag{1}$$

from which the well-known (Klein-Gordon) equation of motion follows

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\Phi = 0 \,. \tag{2}$$

We impose now boundary conditions on the (classical) field Φ. We have to be aware of the importance of the boundary conditions which play an important role because they define the situation and should be imposed in a physical way in a realistic situation[2]. We have chosen the Dirichlet conditions

$$\Phi(t, x = 0) = \Phi(t, x = a) = 0 \,. \tag{3}$$

The complete orthonormal set of functions obeying the boundary problem (2) and (3) with respect to the scalar product related to Eq. (2)

$$(\Phi_1, \Phi_2) := \frac{\mathrm{i}}{\mathrm{c}}\int_0^a \mathrm{d}x \left(\Phi_1^* \partial_t \Phi_2 - \Phi_2 \partial_t \Phi_1^*\right) \,, \tag{4}$$

[1] We are using the notation that space-time interval ds is given by $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu$ and $x^0 = ct$, $x^1 = x$ and greek indices $\in \{0, 1\}$.

[2] For example, in the case of an electromagnetic field, we have the well-known boundary conditions on the surface of an ideal conductor.

is as follows

$$u_n^{(\pm)}(t,x) = \left(\frac{c}{a\omega_n}\right)^{1/2} e^{\pm \mathrm{i}\omega_n t} \sin(k_n x)\,, \tag{5}$$

where

$$\omega_n = \left[\frac{m^2c^4}{\hbar^2} + c^2k_n^2\right]^{1/2}, \qquad k_n = \frac{\pi n}{a}\,, \qquad n \in \{1,2,3,\ldots\}\,.$$

Any solution to our boundary problem can be expanded into the series of functions (5). The canonical quantization of the field Φ is performed by means of such an expansion

$$\Phi(t,x) = \sum_{n=1}^{\infty}\left[u_n^{(+)}(t,x)a_n^+ + u_n^{(-)}(t,x)a_n\right], \tag{6}$$

where a and a^+ are the annihilation and creation operators obeying the canonical commutation relations

$$\left[a_n, a_m^+\right] = \delta_{nm}\,, \qquad [a_n, a_m] = \left[a_n^+, a_m^+\right] = 0\,. \tag{7}$$

The vacuum state $|0\rangle$ is specified as usual by the conditions

$$a_n|0\rangle = 0 \qquad \forall\, n\,. \tag{8}$$

We are interested in the energy of this state, it means we would like to compute the vacuum expectation value of the Hamiltonian density H

$$H = \frac{\partial\mathcal{L}}{\partial(\partial_t\Phi)}\partial_t\Phi - \mathcal{L} = \frac{\hbar c}{2}\left[\frac{1}{c^2}(\partial_t\Phi)^2 + (\partial_x\Phi)^2\right] + \frac{1}{2}\frac{m^2c^2}{\hbar^2}\Phi^2\,. \tag{9}$$

Inserting Eq. (6) into Eq. (9) with respect to Eqs. (7) and Eq. (5), we easily get the energy density in the form

$$\langle 0|H|0\rangle = \frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n - \frac{m^2c^4}{2a\hbar}\sum_{n=1}^{\infty}\frac{\cos(2k_nx)}{\omega_n}\,,$$

so, the total energy of the vacuum state $E(a,m)$ is the integral over $x \in \langle 0, a\rangle$ of the energy density given by the previous formula

$$E(a,m) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n\,. \tag{10}$$

The quantity $E(a,m)$ is evidently divergent but there is a standard possibility to give a meaning to it using a regularization. One of the simplest methods used in the original work [1] is to introduce a dumping function (something like the Boltzmann factor) $\exp(-\epsilon\omega)$ with $\epsilon > 0$ behind the summation sign in Eq. (10). We shall start disscusion about the regularization with the case of a massless field ($m = 0$), because it simplifies the situation. The regularized energy is

$$E_\epsilon(a,0) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\frac{c\pi n}{a}\exp\left(-\frac{c\pi n}{a}\epsilon\right) = \frac{c\pi}{a}\frac{\hbar}{8}\,\frac{1}{\sinh^2\left(\frac{c\pi\epsilon}{2a}\right)}\,. \tag{11}$$

We can expand the quantity $E_\epsilon(a,0)$ into the series of powers of the regularization parameter ϵ. The series contains a term which is singular when ϵ approaches zero.

$$E_\epsilon(a,0) = \frac{c\pi}{a}\frac{\hbar}{8}\left[\frac{a^2}{c^2\hbar^2}\frac{4}{\epsilon^2} - \frac{1}{3} + O(\epsilon^2)\right]. \tag{12}$$

Let us denote by $E^{\mathrm{phys}}(a,0)$ the physical (relevant) energy of our vacuum state. We would like to identify this quantity with the nonsingular term in Eq. (12), i.e. to write

$$E^{\mathrm{phys}}(a,0) = -\frac{\pi\hbar c}{24a}\,, \tag{13}$$

but how to argue that this is the case? One kind of argument could be based on the following. We are not interested in the energy but in the energy differences[3], i.e. the force $F = (E(a+\mathrm{d}a,0) - E(a,0))/\mathrm{d}a$ is important rather than the energy. It means that if we perform several regularizations and renormalizations, we should obtain the same result for the force. We can choose the energy of Minkowski vacuum per length a: ($E^{\mathrm{M}}(a,m)$) to be zero and **define** the physical value of the vacuum energy by

$$E^{\mathrm{phys}}(a,m) := \lim_{\epsilon\to 0^+}\left[E_\epsilon(a,m) - E^{\mathrm{M}}_\epsilon(a,m)\right]. \tag{14}$$

Let us compute $E^{\mathrm{M}}(a,m)$ in the canonical way presented above. The result is

$$E^{\mathrm{M}}(a,m) = \frac{\hbar a}{2\pi}\int_0^\infty \omega\,\mathrm{d}k\,, \quad \text{where} \quad \omega = \left(\frac{m^2c^4}{\hbar^2} + c^2k^2\right)^{1/2}. \tag{15}$$

It is easy to compute the regularized value $E^{\mathrm{M}}_\epsilon(a,m)$ for the massless field. One gets

$$E^{\mathrm{M}}_\epsilon(a,0) = \frac{\hbar a}{2\pi}\int_0^\infty \omega e^{-\epsilon\omega}\mathrm{d}k = \frac{\hbar a}{2\pi c}\int_0^\infty \omega e^{-\epsilon\omega}\mathrm{d}\omega = \frac{\hbar a}{2\pi c}\frac{1}{\epsilon^2}\,. \tag{16}$$

[3]That is always true if the gravitation does not play any role.

Using the definition formula (14) for the physical value of the vacuum state energy, we instantly get above expected result (13).

Macroscopic effect of the vacuum energy is the force (attractive) between the end points of the interval obtained from Eq. (13) as

$$F = -\frac{\mathrm{d}E^{\mathrm{phys}}(a,0)}{\mathrm{d}a} = -\frac{\pi\hbar c}{24a^2}\,.$$

Now we shall rewrite the above described alghoritm into a more convenient and usable form. The renormalized vacuum energy (14) is of the form

$$\lim_{\epsilon\to 0^+}\left[\sum_{n=0}^{\infty} F_\epsilon(n) - \int_0^\infty F_\epsilon(x)\mathrm{d}x\right]\,,$$

where $F_\epsilon(n) = (\hbar/2)(c\pi n/a)f(\epsilon\omega)$, $f(\epsilon\omega)$ is a (smooth) dumping function and F is an analytic function in the complex z half-plane, $\mathrm{Re}(z) > 0$, and the sum and integral exist. Then we can use the so-called Abel-Plana formula (see, e.g. Ref. [5])

$$\sum_{n=0}^{\infty} F_\epsilon(n) - \int_0^\infty F_\epsilon(x)\mathrm{d}x = \frac{1}{2}F_\epsilon(0) + \mathrm{i}\int_0^\infty \frac{F_\epsilon(\mathrm{i}t) - F_\epsilon(-\mathrm{i}t)}{e^{2\pi t}-1}\mathrm{d}t\,. \tag{17}$$

The integral on the right-hand side of Eq. (17) converges uniformly for all $\epsilon > 0$, so we are allowed to perform the limit before the integration. We easily get

$$\begin{aligned} E^{\mathrm{phys}}(a,0) &= \frac{1}{2}0 + \mathrm{i}\int_0^\infty \frac{\frac{\hbar}{2}\frac{c\pi}{a}(\mathrm{i}t) - \frac{\hbar}{2}\frac{c\pi}{a}(-\mathrm{i}t)}{e^{2\pi t}-1}\mathrm{d}t = -\frac{\hbar c\pi}{a}\int_0^\infty \frac{t\mathrm{d}t}{e^{2\pi t}-1} \\ &= -\frac{\hbar c\pi}{a}\frac{1}{(2\pi)^2}\sum_{k=1}^{\infty}\int_0^\infty u e^{-ku}\mathrm{d}u = -\frac{\hbar c\pi}{a}\frac{1}{(2\pi)^2}\frac{\pi^2}{6} \\ &= -\frac{\hbar c\pi}{24a}\,, \end{aligned} \tag{18}$$

that is in accord with (13).

Another way how to obtain the result (13) is to use the renormalization that uses the so-called zeta function regularization (see, e.g., Ref. [4]). The idea is to take the formula (10) and to see that it can be formally understood as follows

$$E(a,0) = \frac{\hbar}{2}\frac{c\pi}{a}\sum_{n=1}^{\infty} n =: \frac{\hbar}{2}\frac{c\pi}{a}\zeta(-1) = E^{\mathrm{phys}}(a,0)\,, \tag{19}$$

where ζ is the Riemann zeta function defined for complex z with $\mathrm{Re}(z) > 1$ by the sum

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \,.$$

The expression $\zeta(-1)$ has to be understood as the value of analytic continuation (see, e.g., Ref. [5]) of the Riemann zeta function into a whole complex plane. Such a continuation is given by the functional formula[4] ([5] or [8])

$$\Gamma\left(\frac{z}{2}\right)\pi^{-z/2}\zeta(z) = \Gamma\left(\frac{1-z}{2}\right)\pi^{(1-z)/2}\zeta(1-z)\,.$$

Using the previous formula, we get $\zeta(-1) = -1/12$, so substituting this value into Eq. (19), we get instantly the result (13).

Now we are going to use the procedure of renormalization based on the Abel-Plana formula (17) for the massive scalar field Φ. We have from (10) and (5)

$$E_\epsilon(a,m) = \frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n e^{-\epsilon\omega_n} = -\frac{mc^2}{4} + \sum_{n=0}^{\infty}\omega_n e^{-\epsilon\omega_n} \quad .$$

The above calculations with $F_\epsilon(x) = \hbar/2\sqrt{m^2c^4/\hbar^2 + \pi^2c^2/a^2x^2}$ lead to the result

$$\begin{aligned} E^{\mathrm{phys}}(a,m) &= -\frac{mc^2}{4} - \frac{\hbar c}{4\pi a}\int\limits_{\lambda}^{\infty}\frac{\sqrt{t^2-\lambda^2}}{e^t-1}\mathrm{d}t = -\frac{mc^2}{4} \\ &- \frac{\hbar c}{4\pi a}\lambda^2\int\limits_{1}^{\infty}\frac{\sqrt{t^2-1}}{e^{\lambda t}-1}\mathrm{d}t \\ &= -\frac{mc^2}{4} - \frac{\hbar c}{4\pi a}\lambda^2\int\limits_{0}^{\infty}\frac{\sinh^2(u)}{e^{\lambda\cosh(u)}-1}\mathrm{d}u\,, \end{aligned} \tag{20}$$

where

$$\lambda = 2\frac{mca}{\hbar}\,.$$

We mention that the constant additive term $-mc^2/4$ does not contribute to the force, and that by putting $m = 0$, we obtain Eq. (12). One can investigate the behaviour of $E^{\mathrm{phys}}(a,m)$ in the limit of a very massive field ($\lambda \to \infty$). It is easy to see that in this case the Casimir's energy and the corresponding Casimir's force are exponentially suppressed by the factor $\mathrm{e}^{-\lambda}$. Let us notice that λ is nothing else but double length of our interval measured in the units of the Compton wavelength of matching particle Φ.

[4] $\Gamma(z)$ is the Euler's gamma function.

3. *Scalar field on a circle*

We consider the scalar field Φ on a circle with the radius equal to $a > 0$. It means that our space-time is a cylinder of radius a (x is the space coordinate, $x \in (0, 2\pi a)$) with time from $-\infty$ to ∞. Equation of motion for such a field is the Klein-Gordon equation (2). The case now differs from the previous one by the boundary conditions imposed on the field. Topology of the circle defines the periodic boundary conditions

$$\begin{aligned} \Phi(t,0) &= \Phi(t,2\pi a)\,, \\ \partial_x \Phi(t,0) &= \partial_x \Phi(t,2\pi a)\,. \end{aligned} \tag{21}$$

Canonical quantization leads to the expansion of the field Φ given by Eq. (6) with the operators a_n and a_n^+ obeying canonical commutation relations (7). Normalized mode functions $u_n^{(\pm)}$ are given by the formulae

$$u_n^{(\pm)}(t,x) = \left(\frac{c}{2a\omega_n}\right)^{1/2} \exp(\pm \mathrm{i}(\omega_n t - k_n x))\,,$$
$$\omega_n = \left[\frac{m^2c^4}{\hbar^2} + c^2k_n^2\right]^{1/2} \quad \text{and} \quad k_n = \frac{n}{a} \quad \text{with} \quad n \in \boldsymbol{Z}\,. \tag{22}$$

Using the expansion (6) with respect to the formulae (22), we can get the unrenormalized energy of the vacuum (once again specified by (8)) in the form

$$E(a,m) = \int_0^{2\pi a} \langle 0|H(t,x)|0\rangle \mathrm{d}x = \frac{\hbar}{2} \sum_{n=-\infty}^{+\infty} \omega_n = -\frac{mc^2}{2} + \hbar \sum_{n=0}^{\infty} \omega_n\,. \tag{23}$$

The sum expressing $E(a,m)$ is again divergent and should be renormalized. We are going to renormalize[5] it using the Abel-Plana formula (17) with

$$F(z) = \frac{\hbar c}{a}\sqrt{\lambda^2 + z^2} \quad \text{where} \quad \lambda = \frac{mca}{\hbar}\,. \tag{24}$$

The result is

$$E^{\mathrm{phys}}(a,m) = -\frac{2\hbar c}{a}\int_\lambda^\infty \frac{\sqrt{t^2-\lambda^2}}{e^{2\pi t}-1}\mathrm{d}t = -\frac{2\hbar c}{a}\lambda^2 \int_0^\infty \frac{\sinh^2(u)}{e^{2\pi\lambda\cosh(u)}-1}\mathrm{d}u\,. \tag{25}$$

The special case is the massless field, which corresponds to the situation λ being zero. In this case, we are able to express E^{phys} in an algebraical form

$$E^{\mathrm{phys}}(a,0) = -\frac{\hbar c}{12a}\,. \tag{26}$$

[5]The meaning of the definition formula (14) in this case is the same as in Sect. 2, because the Minkowskian vacuum now corresponds to the same one, the vacuum of Φ in the two dimensional Minkowski space-time.

We note that at $\lambda \gg 1$ both Casimir's energy and force are again exponentially suppressed by the factor $e^{-\lambda}$ (see Fig.1). This follows from Eq. (25).

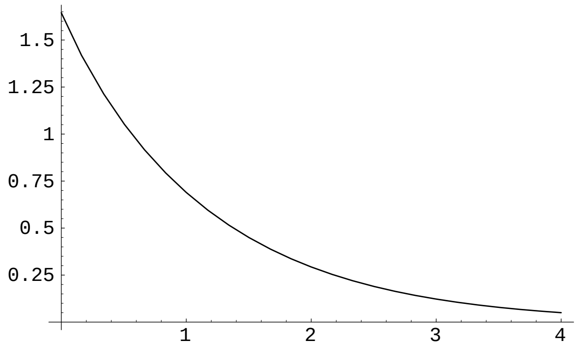

Fig. 1. Typical dependence of the Casimir's energy of a scalar field on both a line segment and a circle on λ *parameter - the function* $f(\lambda) = \int_\lambda^\infty \sqrt{x^2 - \lambda^2}/(\exp(x)-1)$.

4. Scalar field on a two-dimensional sphere

We consider a real scalar field on a two dimensional sphere (S^2) of the radius a. It means that our space-time is $S^2 \times \boldsymbol{R}^1$. It is somewhat more difficult to describe the dynamics of the field than in the previous two sections. It is so due to the fact that our current space-time is curved (its Riemann curvature tensor is not zero). In general, a lot of problems follow from this fact at the quantum level (see, e.g., Ref. [3]). Our point of view will be canonical. We define the dynamics of the field Φ giving the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sqrt{|g|}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - [m^2 - \Xi R]\Phi^2\right] , \tag{27}$$

where $g_{\mu\nu}$ are the components of the metric. In the standard spherical coordinates (θ, ϕ) one has

$$g = c^2 \mathrm{d}t \otimes \mathrm{d}t - a^2\left[\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi \otimes \mathrm{d}\phi\right] \equiv g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu ,$$

where $|g|$ is the absolute value of the determinant of the metric, m is the mass of the field, R is the scalar curvature (computed with respect to the Levi-Civita connection) and Ξ is a coupling constant to the scalar curvature. At the moment, Ξ is not specified.

We shall use the coordinates (t, θ, ϕ) in what follows, so we put down $g = ca^2\sin(\theta)$. The scalar curvature is given by

$$R = -\frac{2}{a^2} .$$

The equation of motion which follows from the Lagrangian density (27) is the covariant wave equation

$$|g|^{-1/2}\partial_\mu\left(|g|^{1/2}g^{\mu\nu}\partial_\nu\Phi\right)+(m^2-\Xi R)\Phi=0\,. \tag{28}$$

In the coordinates we are using, one obtains

$$\frac{\partial^2\Phi}{\partial t^2}-\frac{c^2}{a^2}\Delta_{\theta\phi}\Phi+c^2\left(\frac{m^2c^2}{\hbar^2}+\frac{2\Xi}{a^2}\right)\Phi=0\,, \tag{29}$$

where $\Delta_{\theta\phi}$ is the standard Laplace operator on S^2.

We see that a nonzero value of Ξ corresponds only to the redefinition of the mass. It is so due to the time independence of the metric. In the general case, the coupling to the curvature is very important and leads to many effects (see Ref. [3]). We shall use the so-called conformal coupling $\Xi=1/8$. We can introduce the new (effective) mass M as follows

$$\frac{M^2c^2}{\hbar^2}:=\frac{m^2c^2}{\hbar^2}+\frac{2\Xi}{a^2}\,. \tag{30}$$

Now we have to find a complete orthonormal set of solutions to Eq. (29), say $u^{(\pm)}(t,\theta,\phi)$. One can separate variables in Eq. (29) by putting

$$\Phi(t,\theta,\phi)=\exp(\mathrm{i}\omega t)Y_{lm}(\theta,\phi)\,,$$

where Y_{lm} are the spherical harmonics. They are eigenfunctions of the Laplace operator $\Delta_{\theta\phi}$ on the sphere with eigenvalues $-l(l+1)$ where $l\in\{0,1,2,\ldots\}$ and $m\in\{-l,-l+1,\ldots,l\}$. Substituiting this ansatz to Eq. (29), we get the following set $u^{(\pm)}$ of solutions to our equation

$$\begin{aligned} u^{(+)}_{lm}(t,\theta,\phi) &= \frac{1}{a}\sqrt{\frac{c}{2\omega_l}}\exp(\mathrm{i}\omega_l t)Y_{lm}(\theta,\phi)\,,\\ u^{(-)}_{lm}(t,\theta,\phi) &= (u^{(+)}_{lm}(t,\theta,\phi))^*\,, \end{aligned} \tag{31}$$

which is orthonormal (with respect to the scalar product (4)) and complete (this follows from the properties of the spherical harmonics). The frequency ω_l (as well as energy) depends only on the quantum number l and is given by

$$\omega_l=\left[\frac{m^2c^4}{\hbar^2}+\frac{c^2}{a^2}(2\Xi+l(l+1))\right]^{1/2}=\left[\frac{M^2c^4}{\hbar^2}+\frac{c^2}{a^2}l(l+1)\right]^{1/2}. \tag{32}$$

The field operator Φ can be expanded into the modes $u^{(\pm)}_{lm}$ as follows

$$\Phi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[u^{(+)}_{lm}a^+_{lm}+u^{(-)}_{lm}a_{lm}\right]\,, \tag{33}$$

where a^+ and a are creation and annihilation operators obeying the commutation relations

$$[a^+_{lm}, a_{l'm'}] = -\delta_{ll'}\delta_{mm'} \quad [a_{lm}, a_{l'm'}] = [a^+_{lm}, a^+_{l'm'}] = 0\,.$$

Now we can insert this expansion into the Hamiltonian density, constructed from the Lagrangian density (27) in a canonical way, integrate over all sphere with the result that the unrenormalized energy of the vacuum state is[6]

$$\begin{aligned} E(a,m) &= \frac{\hbar}{2}\sum_{l=0}^{\infty}(2l+1)\omega_l = \hbar\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\omega_l = \\ &= \hbar\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\sqrt{\frac{M^2c^4}{\hbar^2}+\frac{c^2}{a^2}l(l+1)} \\ &= \frac{\hbar c}{a}\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\sqrt{\frac{m^2c^2}{\hbar^2}+\left(l+\frac{1}{2}\right)^2}\,. \end{aligned} \tag{34}$$

Since $E(a,m)$ is divergent, it should be renormalized. We are going to use the Abel-Plana formula (a little modified) to do this. But first, we would like to mention that $M^2 \geq 0$ follows from the formulae (31) and (32) and their physical interpretation. If it were not the case we would get at least one exponentially (in time) expanding and one decreasing mode that do not conserve the probability. One can understand this fact as a constraint to the value of Ξ at the given values of m and a.

The possibility to use something like Abel-Plana formula (17) is based on the fact (we are using now the same logic as in Sect. 2) that the energy of Minkowski vacuum (2+1 Minkowski space) per area of the sphere of radius a ($4\pi a^2$) is given by

$$\begin{aligned} E^{\mathrm{M}}(a,m) &= 4\pi a^2\frac{\hbar}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}k_1\mathrm{d}k_2}{(2\pi)^2}c\sqrt{k_1^2+k_2^2+\frac{m^2c^2}{\hbar^2}} = \\ & a^2\hbar c\int\limits_0^{\infty}\mathrm{d}kk\sqrt{k^2+\frac{m^2c^2}{\hbar^2}} = \frac{\hbar c}{a}\int\limits_0^{\infty}\mathrm{d}zz\sqrt{z^2+\frac{M^2c^2a^2}{\hbar^2}}\,. \end{aligned} \tag{35}$$

Now, combining the last two formulae, we can use the half-integer Abel-Plana

[6] In this formula the following property of spherical harmonics has been used:

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)Y^*_{lm}(\theta,\phi) = \frac{2l+1}{4\pi}$$

formula (see [5])

$$\sum_{n=0}^{\infty} F(n+1/2) - \int_0^{\infty} \mathrm{d}z F(z) = -\mathrm{i} \int_0^{\infty} \mathrm{d}z \frac{F(\mathrm{i}z) - F(-\mathrm{i}z)}{e^{2\pi z}+1} \,,$$

with $F(z) = \hbar c/az\sqrt{z^2+\lambda^2}$, where

$$\lambda = \frac{mca}{\hbar} \,,$$

to get the final result for the Casimir's energy. Computations lead to the result

$$E^{\mathrm{phys}}(a,m) = 2mc^2\lambda^2 \int_0^1 \frac{z\sqrt{1-z^2}}{\exp(2\pi\lambda z)+1} \mathrm{d}z \,. \tag{36}$$

Plots of this function and of its first derivative are shown in Fig. 2.

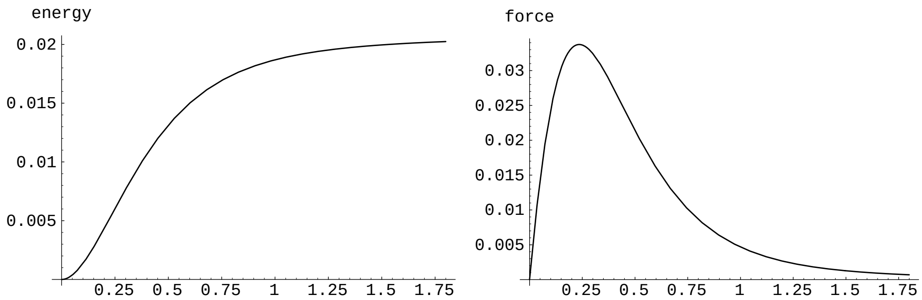

Fig. 2. Casimir's energy and force on the two dimensional sphere. Plots of the function $f(\lambda) = \lambda^2 \int_0^1 z\sqrt{1-z^2}/(\exp(2\pi\lambda z)+1)\mathrm{d}z$ *(energy, left) and its first derivative (absolute value of the force).*

The Casimir's energy on a sphere is not exponentially suppressed at large values of the field mass m, but the force is. So, we obtained the similar result for the behaviour of the Casimir effect at the very large value of the field mass as in the two previous sections. The energy $E^{\mathrm{phys}}(a,m)$ increases as the radius increases, so the Casimir's force makes the sphere small - it is attractive.

5. *Scalar field on a non-commutative two-dimensional sphere*

Since the works of Connes [9], the ideas of the non-commutative (NC) geometry have been applied to physics many times. In the case of three- and two-dimensional models (two space dimensions + external commutative time or one space dimension and one noncommuting time), there are many papers in which the quantum mechanics and the field theory have been formulated. For the review see Ref. [13] and the references therein. Special effects have been investigated in Refs. [11] and [12]. The physical idea to replace our standard point of view on the space-time by the NC geometry is based on the statement that at very short distances, the standard concept of a smooth manifold would change into somewhat else. "The short distances" are usually accepted to be of the Planck length $l_{\rm P}$

$$l_{\rm P} = \left(\frac{\kappa\hbar}{c}\right)^{1/2} \approx 10^{-35} m \,,$$

where $\kappa = 6.67 \times 10^{-11}$ Nm^{2}kg^{-2} is the Newton gravitational constant. One of the main goals of the field theories formulated on noncommutative versions of the standard spaces is that the UV-regularization appears automatically in such theories if the space under consideration is compact (like S^2). This is due to the fact that all operators have finite dimensional representations (they can be written as the matrices of finite rank) and, therefore, there is no place for UV-divergences. But in the case of non-compact spaces, the situation quite differs and UV-divergences persist in the theory [14], and moreover, the new divergences appear. We will discuss this effect for a special case in the last section of this paper.

As we have seen in Sect. 4, the scalar field on the standard, commutative sphere can be expanded into the series (33), where the space-dependence of Φ is encoded in the spherical harmonics $Y_{lm}(\theta, \phi)$. So, for a fixed moment of time, one can write

$$\Phi(t, \boldsymbol{x}) = \Phi(\boldsymbol{x}) = \sum_{lm} \alpha_{lm} Y_{lm}(\boldsymbol{x}) \,,$$

where α_{lm} are complex constants obeying $\alpha_{l-m} = \alpha^*_{lm}$, due to the fact the field is real. It means that the scalar field Φ is a function $S^2 \to \boldsymbol{R}$ at a fixed moment of time. In Ref. [10], the idea is used that the property Φ is defined on S^2 is encoded in the properties of spherical harmonics Y_{lm} that are a functional representation of the rotation group $SO(3)$. The function Φ can be also treated as the function of three real parameters $(x_1, x_2, x_3) \in \boldsymbol{R}^3$ with the constraint $x_1^2 + x_2^2 + x_3^2 = a^2 > 0$. The set of all those functions forms the commutative algebra $\mathcal{A}_\infty$[7].

The noncommutativity of the sphere is introduced putting down the nontrivial commutation relations

$$[\hat{x}_i, \hat{x}_j] = {\rm i}\gamma\epsilon_{ijk}\hat{x}_k \quad i, j, k \in \{1, 2, 3\} \,, \tag{37}$$

where γ is a real fundamental constant characterising the space non-commutativity and operators $\hat{x}_i$ supply the standard cartesian coordinates x_i. The constraint

[7] Algebra is a linear space when a product ".": $\mathcal{A}_\infty \times \mathcal{A}_\infty \to \mathcal{A}_\infty$ is defined and some standard axioms hold. In our case, the algebra's product is nothing else but the pointwise product of real-valued functions, $(\Phi_1.\Phi_2)(x) := \Phi_1(x)\Phi_2(x)$, which is, of course, commutative: $\Phi_1.\Phi_2 = \Phi_2.\Phi_1$ for each pair of functions Φ_1, Φ_2.

defining the sphere radius

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = a^2 Id\,, \tag{38}$$

is considered, too[8]. It is easy to verify that these two relations are not in contradiction. Now we loose the interpretation of $\hat{x}_i$ as the points due to nontrivial commutation relations (37). This situation is well-known from the phase-space picture of quantum mechanics where the Heisenberg uncertainty principle, as a consequence of commutation relation $[x, p_x] = \mathrm{i}\hbar$, removes the idea of pointwise phase space of the classical mechanics. There are no points but only the cells of area $2\pi\hbar$ in which a particle can be localized. By the scalar field $\hat{\Phi}$ on the NC sphere, we shall consider any (operator-valued) function of $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ like in commutative case. The set of all those functions forms again an algebra but now the algebra is non-commutative due to Eq. (37). We denote it by $\mathcal{A}_N$, where the number N should be specified by construction of a representation of our commutation relation.

In Ref. [10], the representation of the commutation relations (37) with the constraint (38) has been done using the Wigner-Jordan realization of the generators $\hat{x}_i$ of our NC algebra. The result is that such a representation is finite-dimensional with dimension N that depends on the values of a and γ,

$$\frac{a}{\gamma} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}\,. \tag{39}$$

Some differential operators, like Laplacian, play an important role in the field theory, especially on the sphere. In our case, we should be interested in Laplacian (squared angular momentum). On the standard sphere, $\Delta = \sum_{i=1}^{3} J_i^2$, where J_i are the anglular momentum operator which may be treated as the Killing's vector fields on the sphere. They act on a function $\Phi \in \mathcal{A}_\infty$ as follows

$$(J_i\Phi)(\boldsymbol{x}) = -\mathrm{i}\epsilon_{ijk}x_j\frac{\partial\Phi}{\partial x_k}(\boldsymbol{x})\,.$$

There are natural analogues $\hat{J}_i$ of the operators J_i in the case of NC algebra $\mathcal{A}_N$ [10]

$$\hat{J}_j\hat{\Phi} := [\hat{X}_i, \hat{\Phi}] \quad \text{, where} \quad \hat{X}_i = \hat{x}_i/\gamma \quad \text{and} \quad \hat{\Phi} \in \mathcal{A}_N\,. \tag{40}$$

The operators $\hat{J}_i$ satisfy the same commutation relations as J_i ($su(2)$ commutation relation), $[\hat{J}_i, \hat{J}_j] = \mathrm{i}\epsilon_{ijk}\hat{J}_k$. In Ref. [10], the eigenfunctions $\hat{Y}_{lm}$ of $\sum_i \hat{J}_i^2 =: \hat{J}^2$ are constructed. They are similar to the spherical harmonics,

$$\hat{J}^2\hat{Y}_{lm} = l(l+1)\hat{Y}_{lm} \quad \text{with} \quad l \in \{0, 1, 2, \ldots, N(a,\gamma)\} \quad m \in \{-l, \ldots, l\}\,.$$

[8] If we considered the relations (37), we would not get in a straighforward way the NC version of the three-dimensional space (the standard configuration space for a free particle). This is due to the fact that the relations (37) are not invariant under space translations which are usually considered to be the fundamental symmetry of the space.

The number $N(a,\gamma)$ is a natural cut-off. Now we can add to our algebra $\mathcal{A}_N$ one external commutative parameter, the time, in accord with the expansion (33). Then the field expansion on the NC sphere, with added commutative time, is of the form

$$\hat{\Phi} = \sum_{l=0}^{N(a,\gamma)} \sum_{m=-l}^{l} \left[e^{\mathrm{i}\omega_l t} \hat{Y}_{lm} a_{lm}^{+} + e^{-\mathrm{i}\omega_l t} \hat{Y}_{lm}^{*} a_{lm} \right] , \tag{41}$$

where $N(a,\gamma)$ is given by (39). So, if we omit all no-interesting parameters (like mass, speed of light), we can write for the energy of ground vacuum state

$$E^{\mathrm{NC}}(a,\gamma) \sim a^2 \sum_{l=0}^{N(a,\gamma)} (2l+1) \sqrt{1 + \frac{1}{a^2} l(l+1)} \, . \tag{42}$$

We see that $E^{\mathrm{NC}}(a,\gamma)$ is finite, so no regularization is needed. The dependence of the Casimir's energy (42) on the sphere radius a is shown in Fig. 3. For a large value of N, i.e., large value of the fraction a/γ, it is almost proportional to a^4, so the Casimir's force, which is also attractive, is proportional to a^3. For small values of the fraction a/γ, this dependence is stair-step, but the Casimir energy never decreases with the sphere radius. So we see that the result is not in accord with the commutative one where the Casimir's force achieves a maximum at the suitable combination of parameters a and m.

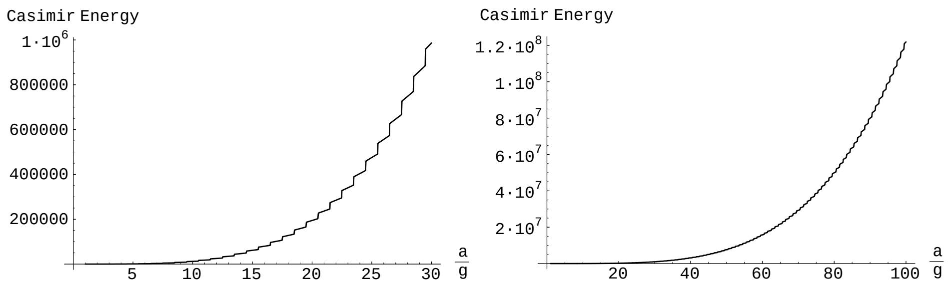

Fig. 3. Casimir's energy on a noncommutative two-sphere.

6. Discussion

In this paper, we have investigated the Casimir effect in various situations. In Sects. 2, 3 and 4, the computations of the Casimir's energy of a scalar field on the standard smooth spaces have been done. Two different ways of appearing of the Casimir effect have been presented: in Sect. 2, the Casimir's energy is caused by the boundaries and in Sect. 3 and Sect. 4 by the topologies of spaces under consideration. These results are known and they or their little modifications may be found in the literature (e.g., see Refs. [2,3] and the references therein). The new result is presented in Sect. 5, where the Casimir's energy on a non-commutative two-sphere has been computed. This case differs from the others because of the finite number of field modes caused by the noncommutativity. So, the Casimir's energy (42) is finite without any renormalization and depends on the radius of the sphere and the parameter of noncommutativity γ. One expects that it would be interesting to investigate what happens as γ tends zero. It is very often quoted that one can obtain the standard (commutative) results from the noncommutative ones in the limit $\gamma = 0$. This statement is true if one is investigating the noncommutative modifications of the quantum mechanics (e.g., see Ref. [15]). We see that this is not the case for the Casimir effect because if we do the limit $\gamma \to 0$ in (42) we will not obtain the result (36). We would get only the infinite expression which could be understood as the starting point for the renormalization procedure. This difference between the features of the Casimir effects on a commutative space and its noncommutative version is known in the case of a noncommutative cylinder [11]. In that case, the space-time is a cylinder times the real time. Noncommutativity appears only on the cylinder (the time is commutative). The noncommutativity does not lead to a natural cut-off of the field modes because the cylinder is not a compact space. So, the computation of Casimir's energy involves a renormalization and regularization as it was done in the above mentioned work. The result (the formula (3.16) of Ref. [11]) is essentially different from the commutative one. In the case of a noncommutative cylinder, the Casimir's energy per unit length does not depend on the cylinder radius r but in the commutative case this quantity depends on r as r^{-2}. There is also another questionable point in the computation of the Casimir's energy according Ref. [11], namely, the cylinder-radius nondependent (divergent) term in Eq. (3.15) of Ref. [11], from which the final result is obtained, contains a logarithmically divergent (in the regularization parameter) part that does not appear in the expression for the regularized energy density of the massless scalar field on the commutative plane. In our opinion, the presence of this term can be explained by the fact that we have to consider the noncommutative plane instead the commutative one to use the formula (14). In this way we have defined the parameter of noncommutativity of the plane but it remains an open question whether this definition of the noncommutativity parameter of the plane is model (mass/type of the field, etc.) independent. Such discrepances might be caused by the fact that our (or of Refs. [11] and [12]) field theories are not built on the space-times with usual symmetries - the time is always added to the noncommutative space as an external commuting parameter which cannot be mixed with others coordinates. In this way, the situation is more similar to the nonrelativistic quantum mechanics than to a relativistic theory. Maybe, one should find another way how to construct a field theory on a noncommutative space as presented above or in Refs. [11] and [12].

Acknowledgements

I am grateful to Peter Prešnajder for useful discussions.

References

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51** (1948) 793.

[2] *Physics in the Making*, ed. A. Sarlemijn and M.J. Sparnaay, North-Holland, Amsterdam (1989).

[3] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Camb. Univ. Press, Cambridge (1982).

[4] N. N. Bogoljubov and D. V. Shirkov, *Quantized Fields* (in russian), Nauka, Moscow (1980).

[5] M. A. Jevgraffov, *Analytic Functions* (in russian), Nauka, Moscow (1968).

[6] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rep. **353** (2001) 1; quant-ph/0106045.

[7] A. A. Starobinsky, Phys. Lett. B **91** (1980) 99; A. Vilenkin, Phys. Rev. D **32** (1985) 32.

[8] I. S. Gradstein and I. M. Ryshik, *Tables of Integrals, Sums, Series and Products*, Nauka, Moscow (1971).

[9] A. Connes, Publ. IHES **62** (1986) 257; A. Connes, *Geometrie Noncommutative*, Inter Editions, Paris (1990).

[10] H. Grosse, C. Klimčik and P. Prešnajder, Int. J. Theor. Phys. **35** (1996) 231; hep-th/9505175.

[11] M. Chaichian, A. Demichev, P. Prešnajder, M. M. Sheikh-Jabbari and A. Tureanu, Nucl. Phys. B **611** (2001) 383; hep-th/0012175.

[12] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. **86** (2001) 2716; hep-th/0010175.

[13] M. Chaichian, A. Demichev and P. Prešnajder, J. Math. Phys. **41** (2000) 1647; hep-th/9904132. H. Grosse, C. Klimčik and P.Prešnajder, hep-th/9510177.

[14] M. Chaichian, A. Demichev and P. Prešnajder, Nucl. Phys. B **567** (2000) 360; hep-th/9812180.

[15] V. P. Nair and A. P. Polychronakos, Phys. Lett. B **505** (2001) 267; hep-th/0011172; A. Hatzinikitas and I. Smyrnakis, hep-th/0103074; M. Demetrian and D. Kochan, Acta Physica Slovaca **52** (1) (2002) 1; hep-th/0102050; O. F. Dayi and A. Jellal, Phys. Lett. A **A287** (2001) 349; cond-math/0103562. J. Gamboa, M. Loewe and J.C. Rojas, Phys. Rev. D **64** (2001) 067901; hep-th/0010220.

[16] G. Bressi, G. Carugno, R. Onofrio and G Ruoso, Phys. Rev. Lett. **88** (2002) 041804; quant-ph/0203002.

CASIMIROV EFECT U ČETIRI SLUČAJA, UKLJUČIVŠI NEKOMUTATIVNU DVOSFERU

Daju se ishodi računa Casimirovog efekta za realno skalarno polje u četiri slučaja: na dužini, na kružnici, te na komutativnoj i nekomutativnoj dvosferi. Glavni je cilj ovog rada raspraviti Casimirovu energiju na nekomutativnoj sferi u okviru teorije s komutativnim vremenom. Raspravlja se također (ne)komutativni valjak.