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FRAMEWORK FOR FINITE ALTERNATIVE THEORIES TO A QUANTUM FIELD THEORY

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The proposed framework is intended to faciliate the search for Lagrangian-based regularizations that introduce only such coefficients which are not regarded as formal auxiliary parameters that will eventually be disposed of during renormalization. Using the path-integral formalism, we put forward a new framework for such finite, alternative quantum theories to a given quantum field theory that have no formal auxiliary parameters. The corresponding alternative, perturbative Green functions are acceptably regularized perturbative expansions of the original Green functions, causal and imply no unphysical free particles. To demonstrate that the proposed framework is feasible, we provide an alternative to the quantum field theory with the ϕ^4 interaction.

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1. Introduction

In this paper, we put forward a framework for constructing such alternative theories to a given quantum field theory (QFT) that provide solutions to the problem of ultraviolet divergencies *without using formal, auxiliary parameters.* For simplicity, we consider in some detail solely the case of the QFT of a single, self-interacting real scalar field $\phi(x)$ with ϕ^4 interaction. In this case, the conventional, QFT, connected, n-point Green functions, scaled by a positive coefficient $Z^{-n/2}$, $G_Z^{(n)}(x_1,\ldots,x_n)$, can be formally expressed as functional derivatives of a generating functional, i.e.,

$$G_Z^{(n)}(x_1,\ldots,x_n) = \left.\frac{(-\mathrm{i})^n\delta^n \ln Z[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}, \tag{1}$$

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with the generating functional

$$Z[J] = \int \prod_x \mathrm{d}\phi(x)\, e^{\mathrm{i}I[\phi,J]}\,. \tag{2}$$

Here the action functional is

$$I[\phi, J] = \int \mathrm{d}^4x \Big(\mathcal{L}_0(\phi, \partial_\mu\phi) - V(\phi) + \phi J\Big)\,, \tag{3}$$

the QFT free-field Lagrangian

$$\mathcal{L}_0 = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\,, \tag{4}$$

and the interaction Lagrangian

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{Z^2\lambda_0 - \lambda}{4!}\phi^4 + \frac{Z-1}{2}(\partial_\mu\phi)^2 + \frac{Zm_0^2 - m^2}{2}\phi^2\,. \tag{5}$$

m^2, m_0^2, λ, and λ_0 are positive coefficients and the external source $J(x)$ is an arbitrary, real scalar field. We use the $(-1, 1, 1, 1)$ metric. An expression for the QFT scattering matrix of the quantum system, characterized by the action functional $I[\phi, J]$, can be obtained in terms of $G_Z^{(n)}$ by means of the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula, see, e.g. Sect. 10.3 in [1].

We can formally rewrite the generating functional (2) as

$$\begin{aligned} Z[J] \quad &= \quad \exp\left\{-\mathrm{i}\int \mathrm{d}^4x\, V(-\mathrm{i}\delta/\delta J)\right\} \\ &\qquad \times \int \prod_x \mathrm{d}\phi(x)\, \exp\Big\{\mathrm{i}\int \mathrm{d}^4x\, \Big(\mathcal{L}_0(\phi, \partial_\mu\phi) + \phi J\Big)\Big\}\,. \end{aligned} \tag{6}$$

Let $\phi_c(x)$ be the Feynman-Stueckelberg solution to the Euler-Lagrange equations of the action $I[\phi, J]$ with $V = 0$. On changing the variable of functional integrals in (6) from ϕ to $\phi - \phi_c$, Eq. (6) can formally be written as

$$\begin{aligned} Z[J] \quad &= \quad N\,\exp\left\{-\mathrm{i}\int \mathrm{d}^4x\, V(-\mathrm{i}\delta/\delta J)\right\} \\ &\qquad \times \exp\left\{\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\, \frac{1}{2}J(x)\Delta_F(x-y)J(y)\right\}\,, \end{aligned} \tag{7}$$

where N is a J-independent factor, irrelevant for the calculation of perturbative expansions of $G_Z^{(n)}$, and the Feynman propagator $\Delta_F(x)$ is defined by the relation

$$\phi_c(x) = \int \mathrm{d}^4 y \, \Delta_F(x-y) J(y) \,, \tag{8}$$

see e.g. Sect. 1.2 and App. B in Ref. [2]. By (8), (3) and (4), the propagator $\Delta_F(x)$ is the Fourier transform of

$$\widetilde{\Delta}_F(k) = \frac{1}{k^2 + m^2 - \mathrm{i}\epsilon} \qquad \text{with} \quad \epsilon \searrow 0 \,. \tag{9}$$

We can obtain the perturbative expansions of $G_Z^{(n)}$ by replacing $Z[J]$ in (1) with (7), and then expanding in powers of the four coupling coefficients λ, $Z^2\lambda_0 - \lambda$, $Z-1$ and $Zm_0^2 - m^2$. These expansions lead to the conventional Feynman rules for this theory [1, 2]. In general, integrals over independent loops diverge, and perturbative expansions of $G_Z^{(n)}$ are not well defined. Regularizing such divergent integrals in perturbative expansions of $G_Z^{(n)}$, and then using a renormalization scheme and removing the cut-off and its auxiliary parameters, we can calculate the perturbative expansion of the QFT scattering matrix by the LSZ reduction formula. There is an infinite variety of *acceptable regularizations*, i.e. such that the above procedure yields the same, renormalization-scheme-dependent, QFT result.

There is a widespread opinion that ultraviolet divergencies encountered in perturbative expansions of Green functions of the standard model are due to *the inadequate description of ultra-high-energy phenomena by the present Lagrangian.* Which suggests that we should modify this Lagrangian to get rid of the ultraviolet divergences. However, as noted, e.g. in Ref. [1], in a QFT, each complete, momentum-space, spin 0 Feynman propagator has the Källén-Lehman spectral representation with a non-negative spectral function, i.e., it is equal to

$$\int_0^\infty \frac{\rho(s)}{k^2 + s - \mathrm{i}\epsilon} \, \mathrm{d}s \tag{10}$$

with some $\rho(s) \geq 0$. That implies that in a QFT, also a complete spin 0 propagator can not vanish faster than $\widetilde{\Delta}_F(k)$ as $|k^2| \to \infty$. So, it follows that in a QFT, we can regularize none of the perturbative expansions of Green functions by modifying its Lagrangian. *Some departure from the conventional framework of QFTs seems necessary to this end, but of course, it has to be compatible with the experimentally verified perturbative results of QFTs.* This presents a sixty-years-old theoretical problem (see, e.g. Sect. 1.3 in Ref. [1] and Ref. [3]).

We believe that there is a Lagrangian-based theory of fundamental interactions such that we can regard perturbative expansions of its Green functions as acceptably regularized perturbative expansions of the Green functions of the standard model. However, up to now each candidate for such a theory that provides a non-perturbative regularization abandons some of the conventional properties of models in contemporary physics. Though upon removal of a cut-off during renormalization, these properties are restored in perturbative results. For example, every version of the Pauli-Villars method proposed so far introduces auxiliary, unphysical particles with negative metric or wrong statistics, lattice regularization is based on discrete space-time and string theories need to introduce additional space-time dimensions to avoid anomalies.

We intend to alleviate this conceptually displeasing situation where models have expected properties only when the cut-off is removed. To this end, we will generalize the 't Hooft-Veltman, Lagrangian-based version of the Pauli-Villars method (HV method), see Sect. 5 in Ref. [4], to a continuous infinity of auxiliary fields but without introducing additional, *auxiliary free particles.* We will *argue* on the analogy of the above path-integral formalism (1)–(9). The key will be the LSZ reduction formula as, in general, *it does not imply a direct connection between the number of free fields and the number of free particles.*

For any given QFT, we introduce a new, Lagrangian-based framework for constructing finite, alternative quantum theories labeled FAQTs that have the following properties: (i) The action functional, I_A, equal the QFT action functional were it not for its free-field Lagrangian. (ii) I_A *is local in space-time and invariant under inhomogeneous Lorentz transformations.* (iii) The connected n-point Green functions, $G_A^{(n)}$, are defined by I_A through a generating functional in analogy to (1). (iv) The scattering matrix, S_A, is defined in terms of $G_A^{(n)}$ by means of the LSZ reduction formula, and *all of its coefficients are regarded as observable quantities.* (v) Free particles, which are determined by the poles of $G_A^{(2)}$, are *the same* as free particles of the corresponding QFT, i.e., there are no additional, unphysical, auxiliary particles. (vi) Perturbative expansions of $G_A^{(n)}$ can be regarded as such acceptably regularized perturbative expansions of the QFT Green functions that are *invariant under the same symmetry transformations as the QFT scattering matrix.* Hence, each FAQT provides a perturbative description at least as adequate as the QFT.

In Sect. 2, we give a short description of the HV method for a single, self-interacting real scalar field in terms of the generating functional. In the HV method, 't Hooft and Veltman (i) introduced a countable set of auxiliary fields, (ii) altered only the QFT free-field Lagrangian and (iii) introduced Feynman-like rules that are the same as the conventional Feynman rules but for an altered propagator, which we label as the HV propagator. We could use the HV method to construct FAQTs were it not for its auxiliary, unphysical free particles.

In Sect. 3, we collate those properties of a HV propagator that are essential for an acceptable covariant regularization. We regard as *an alternative* to $\widetilde{\Delta}_F$ each HV propagator that has these properties but no additional singularities which imply additional, free particles.

In Sect. 4, we give an example of how one may generalize the HV method to a continuous infinity of auxiliary fields so that the corresponding HV propagator is an alternative to $\widetilde{\Delta}_F$. So, we construct a Lorentz invariant FAQT to the QFT of a single, self-interacting real scalar field. Thereby we show that the generalized HV method actually provides a feasible framework for constructing FAQTs to a QFT.

2. *Method of unitary regulators*

In addition to the dimensional regularization, 't Hooft and Veltman invented also the method of unitary regulators (HV method), which is a Lagrangian-based Pauli-Villars method with a discrete spectrum of auxiliary masses (see Sects. 2 and 5–8 in [4]). It seems that the HV method has not been much noticed since we found no reference to it in Science Citation Index, though their report [4] itself has over 200 citations. Here we give an outline of a new *non-perturbative formulation* of the HV method in terms of a generating functional in the case of a single, self-interacting real scalar field with the QFT action functional $I[\phi, J]$.

(a) The HV connected n-point Green functions. Using a countable set of auxiliary real scalar fields $\phi_i(x)$, $i = 1, 2, \ldots$, we define the *HV action functional*

$$I_{HV}[\phi_i, J] = \int \mathrm{d}^4x \left(\mathcal{L}_{HV0}(\phi_i, \partial_\mu\phi_i) - V(\phi_{HV}) + \phi_{HV} J\right), \tag{11}$$

where the HV free-field Lagrangian

$$\mathcal{L}_{HV0} = -\frac{1}{2}\sum_i c_i^{-1}\left((\partial_\mu\phi_i)^2 + \Lambda M_i^2\phi_i^2\right) \quad \text{and} \quad \phi_{HV}(x) = \sum_i \phi_i \,. \tag{12}$$

M_i and c_i are real coefficients such that

$$\Lambda M_1^2 = m^2\,, \quad c_1 = 1\,, \quad \sum_i c_i = 0\,, \quad \sum_i M_i^2 c_i = 0\,, \quad \sum_i |c_i| < \infty\,, \tag{13}$$

and Λ is a positive cut-off parameter. We may regard $I_{HV}[\phi_i, J]$ as a derivative from $I[\phi, J]$ by first replacing the field ϕ with the sum ϕ_{HV} of auxiliary fields ϕ_i, and then modifying I by replacing $\mathcal{L}_0$ with a Lorentz-invariant alteration $\mathcal{L}_{HV0}$. So, in I_{HV} *all fields $\phi_i(x)$ have the same external source* $J(x)$ contrary to QFT formalism.

We define the HV-connected n-point Green functions, $G_{HV}^{(n)}$, through functional derivatives (1) of the generating functional

$$Z_{HV}[J] = \int \prod_{x,i} \mathrm{d}\phi_i(x)\, e^{\mathrm{i}I_{HV}(\phi_i, J)}\,. \tag{14}$$

And we define the HV scattering matrix, S_{HV}, in terms of $G_{HV}^{(n)}$ by means of the LSZ reduction formula (see e.g. Sect. 10.3 in Ref. [1], and Sect. 2.5 in Ref. [4]).

(b) Perturbative expansions. We change the variables of functional integrals in (14) from ϕ_i to $\phi_i - \phi_{ic}$, where $\phi_{ic}(x)$ are solutions to the Euler-Lagrange equations of the action functional I_{HV} with $V = 0$. We then formally obtain

$$\begin{aligned} Z_{HV}[J] &= N_{HV} \exp\left\{-\mathrm{i}\int \mathrm{d}^4x\, V(-\mathrm{i}\delta/\delta J)\right\} \\ &\times \exp\left\{\mathrm{i}\int \mathrm{d}^4x \mathrm{d}^4y\, \frac{1}{2} J(x)\Delta_{HV}(x-y)J(y)\right\}, \end{aligned} \tag{15}$$

where N_{HV} is a J-independent factor, and the *HV propagator* $\Delta_{HV}(x)$ is defined by the relation

$$\sum_i \phi_{ic}(x) = \int \mathrm{d}^4y\, \Delta_{HV}(x-y)J(y)\,. \tag{16}$$

We choose such solutions $\phi_{ic}(x)$ that $\Delta_{HV}(x)$ is the Fourier transform of

$$\widetilde{\Delta}_{HV}(k) = f_{HV}(k^2 - \mathrm{i}\epsilon)\widetilde{\Delta}_F(k) \qquad \text{with} \quad \epsilon \searrow 0\,, \tag{17}$$

where *the regularizing factor* is

$$f_{HV}(z) = 1 + (z + m^2)\sum_{i=2} \frac{c_i}{z + \Lambda M_i^2}\,. \tag{18}$$

By Eqs. (17) and (18), the HV propagator $\widetilde{\Delta}_{HV}(k)$ is a *linear combination* of the Feynman propagators of the HV free-field Lagrangian $\mathcal{L}_{HV0}$, and $\widetilde{\Delta}_{HV} = O(|k^2|^{-3})$ as $|k^2| \to \infty$, by (13).

Expanding expression (15) in terms of the four coupling coefficients λ, $Z^2\lambda_0 - \lambda$, $Z - 1$ and $Zm_0^2 - m^2$ in $V(\phi_{HV})$, we obtain through (1) the perturbative expansions of $G_{HV}^{(n)}$. The resulting expressions lead to the Feynman-like rules for the perturbative calculation of $G_{HV}^{(n)}$. In particular, the perturbative expansions of $G_{HV}^{(n)}$ are calculated by the same sums over the Feynman diagrams as the perturbative expansions of $G_Z^{(n)}$, but with the Feynman propagator $\widetilde{\Delta}_F(k)$ replaced with the HV propagator $\widetilde{\Delta}_{HV}(k)$, whereas the vertices remain unchanged, by (7) and (15). 't Hooft and Veltman [4] obtained such regularized perturbative expansion of $G_Z^{(n)}$ by applying the Feynman rules to the Lagrangian $\mathcal{L}_{HV0} - V$. Due to the form of the interaction and source terms, they could sum up various contributions to yield the above perturbative expansion of $G_{HV}^{(n)}$.

Now let us consider the limiting values of the perturbative expansions of the HV connected Green functions $G_{HV}^{(n)}$ when we remove the regularization by limiting the cut-off parameter $\Lambda \to \infty$. To calculate these limiting values, we: (i) use the Bogoliubov-Parasiouk-Hepp-Zimmermann method, (ii) assume that Z, m_0 and λ_0 are power series in λ, with coefficients that depend on Λ and (iii) determine these coefficients so that, order by order of λ, perturbative expansions of $G^{(n)}_{HV}$ with respect to λ tend towards finite values as $\Lambda \to \infty$ (see e.g. Sect. 2.2 in [2]). Since by (13), the regularization factor f_{HV} is such that

$$\sup_{|z|<z_0} |f^{(n)}_{HV}(z) - \delta_{n0}| \to 0 \quad \text{as} \quad \Lambda \to \infty \quad \text{for any} \quad z_0 > 0, \quad n = 0, 1, \ldots$$

and

$$\sup_{\Lambda>\Lambda_0} \sup_{z>0} |z^n f^{(n)}_{HV}(z)| < \infty \qquad \text{for a} \quad \Lambda_0 > 0, \quad n = 0, 1, \ldots \tag{19}$$

the perturbative expansions of $G^{(n)}_{HV}$ tend towards the corresponding renormalized perturbative expansions of $G^{(n)}_{Z}$ as $\Lambda \to \infty$. As a consequence, the perturbative expansions of $G^{(n)}_{HV}$ are an acceptable regularization of the perturbative expansions of $G^{(n)}_{Z}$.

The perturbative expansions of $G^{(n)}_{HV}$ are causal, free of ultraviolet divergencies, and the corresponding perturbative expansion of S_{HV} is unitary to all orders of the coupling coefficients, but it involves unphysical particles [4]. In particular, the zero-order term in the perturbative expansion of $G^{(2)}_{HV}$ equals the HV propagator $\widetilde{\Delta}_{HV}(k)$. The poles of $\widetilde{\Delta}_{HV}(k)$ determine the properties of free particles of the HV scattering matrix S_{HV}: (i) the position of the pole determines the mass of the particle, and (ii) when the residue is negative, the particle is considered unphysical (a ghost), since the transition probabilities can be negative in the tree order [4]. So, the HV scattering matrix S_{HV} predicts free scalar particles with masses M_i that are unphysical when $c_i < 0$. By (13), the above regularization of $G^{(n)}_{Z}$ by the HV method introduces additional free particles, at least one of which is unphysical.

3. *Alternatives to the QFT propagator*

We do not need the additional poles of the HV propagator $\widetilde{\Delta}_{HV}(k)$ for an acceptable regularization of perturbative expansions of $G^{(n)}_{Z}$. As some of them correspond to unphysical free particles, we would prefer a HV method with such a HV propagator that has properties sufficient for an acceptable regularization of perturbative expansions of $G^{(n)}_{Z}$, but no additional singularities. We will refer to such a HV propagator as an *alternative* to the QFT spin 0 propagator $\widetilde{\Delta}_F(k)$ and denote it by $\widetilde{\Delta}_A(k)$, provided

$$\widetilde{\Delta}_A(k) = f_A(k^2 - \mathrm{i}\epsilon)\widetilde{\Delta}_F(k) \qquad \text{with} \quad \epsilon \searrow 0\,, \tag{20}$$

where: (i) The regularizing factor $f_A(z)$ does not depend on ϵ and is an analytic function of the complex variable z except somewhere along the segment $z \le z_d < -m^2$ of the negative real axis; so $f^*_A(z) = f^*(z)$. (ii) $f_A(-m^2) = 1$. (iii) $f_A(z)$

is real on the positive real axis. (iv) $\sup_z(1 + |z|^{3/2})|f(z)| < \infty$. So all integrals over independent loops converge, and also the Wick rotation is possible. (v) The coefficients of $f_A(z)$ can be made to depend on some cut-off parameter Λ, so that for any $\Lambda \geq \Lambda_0 > 0$, the regularizing factor f_A has the properties (i)–(iv) and satisfies relations (19).

By (i)-(v), we can regard $\widetilde{\Delta}_F$ as a covariant, low-energy approximation to its alternative $\widetilde{\Delta}_A$. The coefficients of $\widetilde{\Delta}_A$ can be chosen to make this low-energy approximation as accurate as desired.

Using Cauchy's integral formula, we can conclude that each $\widetilde{\Delta}_A(k)$ admits the Källén-Lehman representation (10) with the spectral function

$$\rho(s) = \delta(s\ m^2)\pi^{-1}(m^2\ s)^{-1} \lim_{y\to 0} \Im f_A(-s + \mathrm{i}y)\,, \tag{21}$$

$s, y > 0$. By (i)-(iv), $\rho(s)$ is real, $\rho(s) = O(s^{-5/2})$ as $s \to \infty$, and

$$\int_0^\infty s^\ell \rho(s)\,\mathrm{d}s = 0 \qquad \text{for} \qquad \ell = 0, 1; \tag{22}$$

So, $\rho(s)$ is changing sign, in contrast to the spectral function in QFT. By (10) and (21), we may regard $\widetilde{\Delta}_A$ as a sum of $\widetilde{\Delta}_F$ and of a Pauli-Villars regulator with a continuous spectrum of auxiliary masses [5].

Following 't Hooft and Veltman [4], we can show that by replacing the QFT propagator $\widetilde{\Delta}_F$ in the perturbative expansions of $G_Z^{(n)}$ with its alternative $\widetilde{\Delta}_A$, we obtain acceptably regularized, covariant and causal expressions that equal the perturbative expansion of $G_A^{(n)}$. These expressions, by means of the LSZ reduction formula, result in a covariant and unitary perturbative scattering matrix S_A that is without additional free particles but may involve additional, unphysical particles generated by the interaction [4]. However, as there are no additional particles at $V = 0$, there are also no additional particles in a vicinity of $V = 0$ if the zeroes of $(G_A^{(2)})^{-1}$ are continuous functions of the coupling coefficients in this vicinity. However, it is an open question when this is so.

In the next section, we generalize the HV method to auxiliary fields having a continuous, four-vector index so that the sum in the definition (18) of the regularizing factor $f_{HV}(k^2)$ is replaced with an integral that has no poles as a function of k^2. Thereby we avoid the additional free particles implied through the LSZ reduction formula by the additional poles of $\widetilde{\Delta}_{HV}(k)$. So we will be able to demonstrate the existence of such a local and invariant action functional that the corresponding propagator (in the sense of (2)–(7)) is an alternative to $\widetilde{\Delta}_F$. Each such action functional defines a FAQT, as specified in Sect. 1, to the QFT of a single, self-interacting real scalar field.

4. *An example*

One may wonder whether it is possible to generalize the HV method so that the HV propagator $\widetilde{\Delta}_{HV}$ is an alternative to $\widetilde{\Delta}_F$, and thus construct a FAQT to the QFT of a single, self-interacting real scalar field with the action functional (3). The action functional of Eq. (26) demonstrates that this is possible and so provides an *example* of such a FAQT as specified in Sects. 1 and 3.

We introduce infinitely many auxiliary real scalar fields with a continuous, real four-vector index p, $\Phi(x,p)$. Their weighted sum

$$\phi_A(x) = \int \mathrm{d}^4p\, f(p^2)\Phi(x,p)\,, \tag{23}$$

and the FAQT free-field Lagrangian

$$\mathcal{L}_{A0} = \frac{1}{2}\int \mathrm{d}^4p\, \Phi(x,-p)[p^\mu\partial_\mu + t(p^2)]\Phi(x,p) + q\phi_A^2\,, \tag{24}$$

where q is a real coefficient, $f(y)$ and $t(y)$ are real functions of real y, yet to be specified, such that $f^2(y)/t(y)$ is finite at $y=0$, and the integral

$$\int \mathrm{d}^4p \ldots \;\; = -\mathrm{i}\lim_{y\to\infty}\int_{-\mathrm{i}y}^{\mathrm{i}y}\mathrm{d}p^0 \int_{p^2\le y^2}\mathrm{d}^3\mathbf{p}\ldots\;\; . \tag{25}$$

A possible physical significance of such a FAQT free-field Lagrangian is considered in [6].

We construct the FAQT action functional I_A by replacing $\mathcal{L}_0$ with $\mathcal{L}_{A0}$ and ϕ with ϕ_A in the QFT action functional I, i.e.,

$$I_A[\Phi,J] = \int \mathrm{d}^4x\, \left[\mathcal{L}_{A0}(\Phi,\partial_\mu\Phi) - V(\phi_A) + \phi_A J\right]\,. \tag{26}$$

Note that we do not introduce an external source for each individual, auxiliary real scalar field $\Phi(x,p)$, but only for the combination ϕ_A that appears in the interaction Lagrangian $-V$. The FAQT action functional I_A is *real, quadratic, local in spacetime, and invariant under the inhomogeneous Lorentz transformations*

$$x\to Lx+a\,,\quad p\to Lp\,,\quad \Phi(x,p)\to\Phi(Lx+a,Lp)\,,\quad J(x)\to J(Lx+a)\,. \tag{27}$$

We define the FAQT connected n-point Green functions, $G_A^{(n)}$, by functional derivatives (1) of the generating functional

$$Z_A[J] = \int\prod_{x,p}\mathrm{d}\Phi(x,p)e^{\mathrm{i}I_A[\Phi,J]}\,. \tag{28}$$

We also define the FAQT scattering matrix, S_A, in terms of $G_A^{(n)}$ by means of the LSZ reduction formula.

We change the variables of the functional integrals in (28) from $\Phi(x,p)$ to $\Phi(x,p) - \Phi_c(x,p)$, where $\Phi_c(x,p)$ is a solution to the Euler-Lagrange equations of the FAQT action I_A with $V = 0$. So, we can formally write the generating functional (28) as

$$\begin{aligned} Z_A[J] \quad &= \quad N_A \exp\left\{-\mathrm{i}\int \mathrm{d}^4x\, V(-\mathrm{i}\delta/\delta J)\right\} \\ &\qquad \times \exp\left\{\mathrm{i}\int \mathrm{d}^4x \mathrm{d}^4y\, \frac{1}{2} J(x)\Delta(x-y)J(y)\right\}, \end{aligned} \tag{29}$$

where N_A is a J-independent factor, and the propagator $\Delta(x)$ is defined by the relation

$$\int d^4p f(p^2)\Phi_c(x,p) = \int d^4y\, \Delta(x-y)J(y)\,. \tag{30}$$

According to Ref. [7], we can choose such a solution $\Phi_c(x,p)$ that the Fourier transform of $\Delta(x)$ is

$$\widetilde{\Delta}(k) = \frac{-I(k^2 - \mathrm{i}\epsilon)}{2qI(k^2 - \mathrm{i}\epsilon) + 1} \qquad \text{with} \quad \epsilon \searrow 0\,, \tag{31}$$

where $I(z)$ is the analytic function of the complex variable z equal to

$$2\pi^2 \int_0^\infty \mathrm{d}y\, f^2(y)[y/t(y)] \left(\sqrt{1 + zy/t^2(y)} + 1\right)^{-1} \qquad \text{for} \quad z > 0\,. \tag{32}$$

We can choose such q, $f(y)$ and $t(y)$ that the propagator $\widetilde{\Delta}(k)$ becomes an alternative to $\widetilde{\Delta}_F$, and so the action functional I_A defines such a FAQT as specified in Sects. 1 and 3. To show by a simple example that this is feasible: (i) Let

$$t(y) = (-1)^j v_j \sqrt{y} \qquad \text{for} \qquad y \in [j-1, j)\,, \quad j = 1,2,3,4, \tag{33}$$

where $v_j = [1 + (2j-5)\eta)]\Lambda$, $\Lambda \geq \Lambda_0 > m/(1-3\eta)$ and $\eta \in (0, 1/3)$. (ii) Let

$$q = \frac{-m^2 r_1}{2r_1 + m^2 r_{-1}}\,, \qquad \text{with} \qquad r_n = \sum_{j=1}^{4} d_j (v_j^2 - m^2)^{n/2} \tag{34}$$

and $d_j = (-1)^j(4 - |2j-5|)$. (iii) Let $f(y)$ be such that

$$2\pi^2 \int_{j-1}^{j} f^2(y)\sqrt{y}\,\mathrm{d}y = \frac{|d_j| m^2}{2qr_1}\,, \quad j = 1,2,3,4\,, \tag{35}$$

and let $f(y) = 0$ for $y > 4$. In such a case, we can check by inspection that:

(i)

$$\widetilde{\Delta}(k) = f_A(k^2 - \mathrm{i}\epsilon)\widetilde{\Delta}_F(k)\,, \tag{36}$$

with the regularization factor

$$f_A(z) = \frac{\sum_{j=1}^4 d_j(\sqrt{v_j^2 + z} + v_j)^{-1}}{2q\sum_{j=1}^4 d_j(\sqrt{v_j^2 - m^2} + v_j)^{-1}(\sqrt{v_j^2 + z} + v_j)^{-1}(\sqrt{v_j^2 + z} + \sqrt{v_j^2 - m^2}\ )^{-1}}\,. \tag{37}$$

(ii) $f_A(z)$ is an analytic function of z except along the segment $z \leq z_d$, $z_d = -\Lambda^2(1-3\eta)^2 < -m^2$ of the negative real axis, for each $\eta \in (1/3 - 0.047, 1/3)$. To infer this, we take into account that $\Re\sqrt{z} \geq 0$ and that $\Lambda/(\sqrt{v_j^2 - m^2} + v_j) \to \infty$ as $\eta \nearrow 1/3$ only if $j = 1$.

(iii) $f_A(z)$ is real for $z \geq -v_1^2$, $f_A(-m^2) = 1$, $\sup_z(1+|z|^{3/2}) < \infty$, and relations (19) are satisfied. Thus, $\widetilde{\Delta}(k)$ is an alternative to $\widetilde{\Delta}_F(k)$, with $\Lambda \geq \Lambda_0$ as a cut-off parameter.

Note that there are infinitely many functions $f(y)$ and $t(y)$ that result in the same alternative, Eqs. (36) – (37), to the propagator $\widetilde{\Delta}_F(k)$.

In analogy with alternatives to the spin 0 Feynman propagator, we have defined also alternatives to the spin $\frac{1}{2}$ Feynman propagator and to the Feynman propagator in unitary gauge for massive spin 1 bosons, and provided an example of the corresponding FAQT free-field Lagrangians [8]. We used them to construct an example of a FAQT to QED with massive photons in the unitary gauge.

Similarly, we can demonstrate that there are FAQTs to the standard model. However, it is an open question what properties the free-field Lagrangian of a FAQT to the standard model must have to make this theory: (i) without additional particles created by the interaction, (ii) of interest for the study of non-perturbative phenomena, cf. [9], and (iii) a better way of extracting physical data from the interaction Lagrangian of the standard model. In this connection it is not of primary importance how convenient are the resulting, acceptably regularized Green functions for further calculations—the main concern of all regularizations so far, cf. [10].

5. *Some comparisons between the FAQT and QFT formalisms*

From the preceding examples, we note the following differences and similarities between the FAQT and QFT formalisms:

(a) Källén-Lehman representation (10) with the spectral function (21) of the FAQT propagator $\widetilde{\Delta}_A$ defined by (31) is a consequence of assumptions about the

FAQT action functional I_A. These classical assumptions do not imply the QFT premises on which the Källén-Lehman representation (10) with a non-negative spectral function is based in QFT, and do not contradict the positivity postulates of quantum mechanics, cf. Sect. 10.7 in Ref. [1].

(b) According to Buchholz and Haag [12], one can take as a postulate that each QFT "is completely described by a finite number of covariant fields (each having a finite number of components)". However, in constructing a FAQT, we replace each QFT field with a field that has a continuous infinity of components – an essential difference characteristic also of string theories.

(c) If we replace the FAQT free-field Lagrangian $\mathcal{L}_{A0}$ in the FAQT action functional I_A with the QFT free-field Lagrangian $\mathcal{L}_0$ with $\phi = \phi_A$, the resulting action functional becomes equivalent to the QFT action functional I as far as the related Green functions are concerned. So we may take that QFT of a single, self-interacting real scalar field and the FAQT based on I_A actually differ only in their free-field Lagrangian densities. The interaction and source parts of the FAQT action functional I_A are completely determined by the corresponding parts of the QFT action functional I.

(d) Contrary to the QFT free-field Lagrangian, *a FAQT free-field Lagrangian by itself does not specify the free particles of a FAQT*, and the number of FAQT free particles is finite whereas the number of FAQT free fields is not.

6. *Summary*

In this paper: (i) we have put forward a framework for constructing such *finite alternative quantum theories* (FAQTs) to a QFT that respect *the conventional premises of theoretical physics*, and (ii) we have given an example to prove that this framework is feasible.

The basic premises of the proposed mathematical framework for constructing a FAQT to a given QFT are that:

(i) A FAQT can be characterized by a local, invariant action functional.

(ii) The FAQT, connected n-point Green functions are defined by functional derivatives of a generating functional. They are invariant under the same symmetry transformations as the QFT scattering matrix.

(iii) The FAQT scattering matrix is defined in terms of the FAQT Green functions by means of the LSZ reduction formula. Its coefficients are regarded as observable quantities: it has *no auxiliary parameters.*

(iv) The FAQT free particles are the same as the QFT ones.

(v) Perturbative expansions of the FAQT Green functions may be regarded as such acceptably, covariantly regularized perturbative expansions of the QFT Green functions that do not depend on auxiliary parameters.

So each FAQT contains all perturbative results of the QFT in question. In particular, in the limit of a negligible cut-off, the perturbative expansion of the FAQT scattering matrix equals that of the QFT scattering matrix.

We construct a FAQT to a QFT as follows:

(i) We take the QFT Lagrangian and introduce external sources to form the action functional we need to specify its QFT related Green functions through the generating functional.

(ii) In the QFT action functional, we replace each field but the external sources with an average over a continuous infinity of auxiliary fields of the same kind.

(iii) We replace the QFT free-field Lagrangian with such a local free-field Lagrangian that the resulting action functional becomes a FAQT action functional as specified above.

In such a case, the Feynman-like rules for calculating the FAQT Green functions are similar to the ones of the QFT in question. Only the QFT propagators are replaced with FAQT propagators, which are determined through the FAQT action functional without the interaction part.

By providing an example in terms of functions defined in the eight-dimensional space $\mathrm{R}^{1,3} \times \mathrm{R}^{1,3}$, we have shown in Sect. 4 that the above framework, based on the path-integral formalism, is feasible.

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POSTAVKA KONAČNE INAČNE TEORIJE KVANTNOJ TEORIJI POLJA

Predlažu se postavke čiji je cilj olakšanje traženja regularizacija uz primjenu samo onih koeficijenata koji se ne smatraju formalnim pomoćnim parametrima, a koji se kasnije odbacuju renormalizacijom. Rabeći formalizam integrala po putevima, izlažemo novu postavku za takve konačne, inačne kvantne teorije danoj kvantnoj teoriji polja koja nema formalne pomoćne parametre. Odgovarajuće inačne perturbativne Greeenove funkcije su prihvatljivo regulirani perturbativni razvoji izvornih Greenovih funkcija, kauzalni su i ne uključuju nefizikalne slobodne čestice. Radi pokazivanja da je predložena postavka ostvariva, izlažemo inačicu kvantnoj teoriji polja s ϕ^4 interakcijom.