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GLOBAL MONOPOLE IN EINSTEIN-CARTAN THEORY BASED ON LYRA GEOMETRY

FAROOK RAHAMAN[a,b,*] and SAMIR MAL[b]

[a]*Khodar Bazar, Baruipur, Kolkata 700144, West Bengal, India*
[*]*E-mail address: farook_rahaman@yahoo.com*

[b]*Department of Mathematics, Jadavpur University, Kolkata 700032, India*

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We present an approximate solution of Einstein - Cartan equations for the metric outside a monopole resulting from the breaking of a global O(3) symmetry based on Lyra's modified Riemannian geometry. It is interesting to note that unlike the general relativity case, the global monopole in Einstein - Cartan theory based on Lyra's geometry can have attractive as well as repulsive gravitational effect.

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1. Introduction

It is argued that the early universe had undergone a number of phase transitions as it cooled down from the hot initial phase. One of the immediate consequences of these phase transitions is the formation of defects or mismatches in the orientation of the Higgs field in causally disconnected regions [1]. The type of defects depends on the topology of the vacuum manifold M. In particular, monopoles are formed when M contains surfaces which cannot be shrunk to a point, i.e., when $\pi_2(\mathrm{M}) \neq I$. These monopoles have Goldstone fields with their energy density decreasing with distance as r^{-2}. The monopole exerts practically no gravitational force on relativistic matter, but the space around it has a deficit angle. Much work on the issue of global monopoles has been carried out since Barriola and Vilenkin [2] first suggested that these objects were by-produces of a global O(3) spontaneous symmetry breaking down to U(1) [3].

In last few decades, there has been a considerable interest in alternative theories of gravitation because it seems that gravity is not given by the Einstein action. For this reason, different attempts have been carried out to study the gravitational theories other than the one of Einstein.

Lyra [4] suggested a modification of Riemannian geometry which may also be considered as a modification of Weyl's geometry. In Lyra's geometry, Weyl's concept of gauge, which is essentially a metrical concept, is modified by the introduction of a gauge function into the structureless manifold. In consecutive investigations, Sen [5] and Dunn [6] proposed a new scalar-tensor theory of gravitation and constructed an analogue of the Einstein field equations based on Lyra's geometry which in normal gauge may be written as

$$R_{ab} - \frac{1}{2} g_{ab} R + \frac{3}{2} \phi_a \phi_b - \frac{3}{4} g_{ab} \phi_c \phi^c = -8\pi T_{ab} \,, \tag{1}$$

where ϕ_a is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

In a recent work [7], Farook Rahaman has studied the global monopole based on Lyra geometry.

The extension of the geometric principles of general relativity to the physics at a microscopic level, where matter formation is done by elementary particles, characterized by a spin angular momentum in addition to the mass, is achieved in Einstein-Cartan theory [8]. Since spin is a very important property of a particle, it is very relevant to consider its role in the study of the configurations of topological defects.

In this work, we shall deal with monopole in Einstein-Cartan theory based on Lyra's geometry in normal gauge, i.e., displacement vector $\phi_a = (\beta, 0, 0, 0)$, where β is a constant, and look forward whether the monopole shows any significant properties in the above consideration.

2. *Basic equations*

Since the space-time of global monopole is static and spherically symmetric, the metric is taken as

$$\mathrm{d}s^2 = \mathrm{e}^{\gamma} \mathrm{d}t^2 - \mathrm{e}^{\mu} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega_2^2 \,, \tag{2}$$

where μ and γ are functions of r only.

In the original work of Barriola and Vilenkin, the energy momentum tensor due to the monopole field outside the core is [2]

$$T_t^t = T_r^r = \frac{\eta^2}{r^2} \qquad \text{with} \qquad T_\theta^\theta = T_\phi^\phi = 0 \,, \tag{3}$$

where η is the symmetry breaking scale of the theory.

Following Prasanna [9], the Einstein - Cartan equations can be written as

$$R_{ab} - \frac{1}{2} R \delta_a^b = -8\pi G T_{ab} \,, \tag{4}$$

$$Q^a_{bc} - \delta^a_b Q^l_{lc} - \delta^a_c Q^l_{bl} = -8\pi G S^a_{bc} \,. \tag{5}$$

We assume that the spins of the particles composing the monopoles are all aligned in the r-direction. So, the only nonzero component of the spin tensor S_{ab} is $S_{23} = K$. Here S^a_{bc} is the spin density described through the relations

$$S^a_{bc} = U^a S_{bc} \qquad \text{with} \qquad U^c S_{bc} = 0\,, \tag{6}$$

where U_a is the four-velocity, $U^a = \delta^a_4$ and Q^a_{bc} is the torsion tensor given by Eq. (5).

Therefore, from Eqs. (1), (3), (4) and (5), the field equations for the metric (2) in Einstein-Cartan theory based on Lyra's geometry are

$$\mathrm{e}^{-\mu}\left(\frac{\gamma^1}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} - \frac{3}{4}\beta^2 \mathrm{e}^{-\gamma} + 16\pi^2 G^2 K^2 = \frac{8\pi G\eta^2}{r^2}\,, \tag{7}$$

$$-\mathrm{e}^{-\mu}\left(\frac{\mu^1}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} + \frac{3}{4}\beta^2 \mathrm{e}^{-\gamma} + 16\pi^2 G^2 K^2 = \frac{8\pi G\eta^2}{r^2}\,, \tag{8}$$

$$\mathrm{e}^{-\mu}\left(\frac{\gamma^{11}}{2} + \frac{\gamma^{12}}{4} - \frac{\mu^1\gamma^1}{4} + \frac{\gamma^1 - \mu^1}{2r}\right) - \frac{3}{4}\beta^2 \mathrm{e}^{-\gamma} + 16\pi^2 G^2 K^2 = 0\,. \tag{9}$$

3. *Solutions*

Now, subtracting Eq. (7) from Eq. (9) and multiplying by $1/r$, we get

$$\mathrm{e}^{-\mu}\left[\left(\frac{\gamma^{11}}{r} - \frac{\gamma^1}{r^2} - \frac{2}{r^3}\right) - \mu^1\left(\frac{\gamma^1}{r} + \frac{1}{r^2}\right) + \gamma^1\frac{\gamma^1 + \mu^1}{2r}\right] + \frac{2}{r^3} = -\frac{16\pi G\eta^2}{r^3}\,. \tag{10}$$

Differentiating Eq. (7) with respect to r, we obtain

$$\mathrm{e}^{-\mu}\left[\left(\frac{\gamma^{11}}{r} - \frac{\gamma^1}{r^2} - \frac{2}{r^3}\right) - \mu^1\left(\frac{\gamma^1}{r} + \frac{1}{r^2}\right)\right] + \frac{3}{4}\gamma^1\beta^2 \mathrm{e}^{-\gamma} + \frac{2}{r^3} = -\frac{16\pi G\eta^2}{r^3}\,. \tag{11}$$

From Eqs. (10) and (11), it follows

$$\gamma^1\left[\left(\mathrm{e}^{-\mu}\,\frac{\mu^1 + \gamma^1}{2r}\right) - \frac{3}{4}\mathrm{e}^{-\gamma}\beta^2\right] = 0\,. \tag{12}$$

From Eqs. (7) and (8), we get

$$\left[\left(\mathrm{e}^{-\mu}\,\frac{\mu^1 + \gamma^1}{2r}\right) - \frac{3}{4}\mathrm{e}^{-\gamma}\beta^2\right] = 0\,. \tag{13}$$

This is leading to

$$\gamma^1 \neq 0\,. \tag{14}$$

This indicates the existence of the gravitational force. Also, as $\beta \neq 0$, we never get Barriola-Vilenkin–like solutions.

At this stage, we shall solve the field equations (7) - (9) in the weak-field approximation.

So we take the metric coefficients of the form

$$\mathrm{e}^{\mu} = 1 + f(r) \qquad \mathrm{e}^{\gamma} = 1 + g(r) \,. \tag{15}$$

Here functions f and g should be computed to the first order of η^2, β^2 and K^2.

With this approximation, Eqs. (7), (8) and (9) take the following forms

$$-\frac{f}{r^2} + \frac{g^1}{r} - \frac{3}{4}\beta^2 + 16\pi^2 G^2 K^2 = \frac{8\pi G \eta^2}{r^2} \,, \tag{16}$$

$$\frac{f^1}{r} + \frac{f}{r^2} - \frac{3}{4}\beta^2 - 16\pi^2 G^2 K^2 = -\frac{8\pi G \eta^2}{r^2} \,, \tag{17}$$

$$\frac{1}{2} g^{11} + \frac{g^1 - f^1}{2r} - \frac{3}{4}\beta^2 + 16\pi^2 G^2 K^2 = 0 \,. \tag{18}$$

Solving these equations, we get

$$f = \frac{1}{4}\beta^2 r^2 + \frac{16}{3}\pi^2 G^2 K^2 r^2 - 8\pi G \eta^2 \,, \tag{19}$$

$$g = \frac{1}{2}\beta^2 r^2 - \frac{16}{3}\pi^2 G^2 K^2 r^2 \,. \tag{20}$$

Thus, in the weak field approximation, the monopole metric in Einstein-Cartan theory based on Lyra's geometry takes the following form

$$\begin{aligned} \mathrm{d}s^2 = & \left[1 + \frac{1}{2}\beta^2 r^2 - \frac{16}{3}\pi^2 G^2 K^2 r^2\right] \mathrm{d}t^2 \\ & - \left[1 + \frac{1}{4}\beta^2 r^2 + \frac{16}{3}\pi^2 G^2 K^2 r^2 - 8\pi G \eta^2\right] \mathrm{d}r^2 - r^2 \mathrm{d}\Omega_2^2 \,. \end{aligned} \tag{21}$$

4. *Gravitational effects on test particles*

Let us now consider a relativistic particle of mass m, moving in the gravitational field of a monopole described by Eq. (21) and using the formalism of Hamilton and Jacobi (H-J).

The H-J equation is given by [8]

$$\frac{1}{B(r)}\left(\frac{\delta S}{\delta t}\right)^2 - \frac{1}{A(r)}\left(\frac{\delta S}{\delta r}\right)^2 - \frac{1}{r^2}\left(\frac{\delta S}{\delta \theta}\right)^2 - \frac{1}{r^2 \sin^2\theta}\left(\frac{\delta S}{\delta \phi}\right)^2 + m^2 = 0 \,, \tag{22}$$

where $B(r) = 1 + \frac{1}{2}\beta^2 r^2 - \frac{16}{3}\pi^2 G^2 K^2 r^2$ and $A(r) = 1 + \frac{1}{4}\beta^2 r^2 + \frac{16}{3}\pi^2 G^2 K^2 r^2 - 8\pi G\eta^2$.

In order to solve the particle differential equation, let us use the separation of variables for the H-J function S as follows [8]

$$S(t, r, \theta, \phi) = -Et + S_1(r) + S_2(\theta) + J\phi \,. \tag{23}$$

Here the constants E and J are identified as the energy and angular momentum of the particle.

The radial velocity of the particle is (for detailed calculations, see Ref. [10])

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{B}{E\sqrt{A}} \left(\frac{E^2}{B} + m^2 - \frac{p^2}{r^2} \right)^{1/2} , \tag{24}$$

where p is the separation constant. The turning points of the trajectory are given by $\mathrm{d}r/\mathrm{d}t = 0$, and as a consequence, the potential curves are

$$\frac{E}{m} = \sqrt{B} \left(\frac{p^2}{m^2 r^2} - 1 \right)^{1/2} , \tag{25}$$

Thus,

$$\frac{E}{m} = \left[1 + \frac{1}{2}\beta^2 r^2 - \frac{16}{3}\pi^2 G^2 K^2 r^2 \right]^{1/2} \left(\frac{p^2}{m^2 r^2} - 1 \right)^{1/2} . \tag{26}$$

In this case, the extrema of the potential curve are the solutions of the equation

$$\left[\frac{1}{2}\beta^2 - \frac{16}{3}\pi^2 G^2 K^2 \right] m^2 r^4 + p^2 = 0 \,. \tag{27}$$

This equation has a real solution if $\frac{1}{2}\beta^2 < \frac{16}{3}\pi^2 G^2 K^2$. So, the orbit of a massive particle can be trapped by the monopole. In other words, the global monopole exerts attractive gravitational force on the surrounding matter.

If $\frac{1}{2}\beta^2 > \frac{16}{3}\pi^2 G^2 K^2$, then the equation has no real solution. So, orbit of a massive particle can not be trapped by the monopole, i.e., the global monopole exerts repulsive gravitational force on the surrounding matter.

5. *Concluding remarks*

We see that in going from general relativity to Einstein-Cartan theory based on Lyra's geometry, both space-time curvature and topology are affected by the presence of the spin tensor. By studying the motion of test particle, we have shown that the global monopole can have an attractive as well as a repulsive gravitational effect on the matter around it. This example is in striking contrast with the corresponding result in general relativity, as we know that the monopole exerts no gravitational force [2]. From this we conclude that the spins of the particles take an important role for monopole configuration. So, for future work, one can study other topological defects in Einstein-Cartan theory based on Lyra's geometry.

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GLOBALNI MONOPOLI U EINSTEIN-CARTANOVOJ TEORIJI ZASNOVANOJ NA LYRINOJ GEOMETRIJI

Predstavljamo približno rješenje Einstein-Cartanovih jednadžbi za metriku izvan monopola, koja slijedi iz lomljenja globalne O(3) simetrije, zasnovane na Lyrinoj promjeni Riemannove geometrije. Zanimljivo je istaknuti da, za razliku od slučaja opće teorije relativnosti, globalni monopol u Einstein-Cartanovoj teoriji zasnovanoj na Lyrinoj geometriji može gravitacijski privlačiti kao i odbijati okolne mase.