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EQUATION OF STATE FOR PHOTONS ADMITTING TO TSALLIS STATISTICS

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By allowing the distribution of states for a harmonic oscillator to admit to a multifractal distribution at high frequencies, we demonstrate that the spectral distribution of energy for photons admitting the modified statistics has a high energy anomalous behaviour and the equation of state of a photon gas is also modified.

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1. *Introduction*

One of the most sacred and yet rather mysterious principles of quantum theory is the spin statistics connection [1,2]. Geroch and Horowitz [3] have suggested that the exclusion principle is a result of topological properties in spin space and has a basis independent of its derivation from quantum-field theoretic considerations. In this regard the spin – statistics connection can be derived for a relativistic quantum field theory assuming the field operators commute over space-like intervals and with the added restriction imposed by the commutation and anti-commutation relations [4–6]. If, however, particles retain some memory of their collisions, or if long-range interactions are present, both the Boltzmann-Gibbs theory and Bose and Fermi statistics can be violated. To accommodate the existence of a non-Markovian memory in particle collisions, Tsallis [7] has invented a new statistics based on multifractality and scale invariance. Unlike the modified statistical approach of Haldane [8] and Medvedev [9], it has its basis rooted in notions of non-linearity and self-similarity. Applications to the solar plasma [10–12], a generalized H theorem [13–15], the fluctuation dissipation theorem [16], the Langevin and Fokker-Planck equation [17], the equipartition theorem [18], the Ising chain [19,20], paramagnetic systems [21] and the Planck radiation law [22] have led to modifications of the conventional theory that await experimental confirmation. Limits on the non-extensive statistics parameter in Tsallis statistics can be set by finding out how the new statistics affects the primordial helium abundance in cosmology [23]. We have also applied Tsallis statistics to a two-level system [24], a two-state paramagnetic system [25,26], the Debye theory of specific heats [27] and the spectral distribution of photons in black body radiation [28]. As emphasized in Ref. [22], non-extensive statistics applies whenever the linear size of the system is smaller than or comparable to the range of the relevant interaction between the elements of the system. In this regard, it is reasonable to think that the present cosmic microwave background radiation might be slightly different than the Planck distribution due to the long-range gravitational influence. This influence could be a small long-range memory of times when matter and light were still strongly coupled leading to a correlation between the energy levels at a given frequency that exhibits these long-range memory effects and does not generate a distribution of the usual Bose-Einstein form. In the present note, we continue the study of Ref. [28] by allowing the non-extensive parameter to depend on frequency. In Ref. [28], we applied the q statistics of Tsallis to a bosonic oscillator, which might suggest an inconsistency since Abe [29] and Tsallis [30] had explicitly related the q statistics to the q deformed commutation relations. However, as discussed by Tirnakli et al. [31], a generalized q statistics can be thought of as independent of the bosonic, fermionic, or intermediate statistics of the particles involved and can be applied to the energy levels of the system exclusively. Here the statistics of particles is put in at another stage. In the present discussion, we do not use the q expectation value which involves lumping together the q form of the entropy and the deformed commutation relations. Instead, we apply the q (Tsallis) modified entropy with the conventional form of the expectation value of the energy as discussed in Ref. [31]. This point was previously made in a paper on the anharmonic oscillator and Tsallis statistics [32]. Since we assume that a slightly non-extensive behaviour exists for the microwave background due to long-range memory of times when matter and light were still strongly coupled, we would have a natural tendency to believe that these memory effects are more pronounced for high frequencies where matter and radiation were more strongly coupled. This assumption would lead to a dependence of q on the frequency of the mode. Since any curve locally can be fit to a parabolic form, we assume for the dependence of $q-1$ on ν a parabolic form with no linear term. Below the value $h\nu = kT$, we assume that the modes conform to Boltzmann-Gibbs statistics with $q = 1$, since here the memory effects would most likely be insignificant. For a frequency dependence of the non-extensive parameter $\alpha = q - 1 = \alpha_0 + \alpha_1\nu^2$ ($\alpha_0 > \alpha_1$), we find the cut-off as a function of temperature. We then calculate the total energy and pressure of the modified photon gas. Our calculations lead to a dependence of the energy density on the temperature as $U = \alpha T^4 - \gamma T^6$ and a corresponding modified pressure. Such results would modify the cosmological evolution of a photon gas during the radiation era and also lead to anomalies in the spectral emission of condensed astrophysical objects that emit in the X-ray and γ-ray region.

2. Equation of state for photons admitting to Tsallis statistics

We begin by considering the expression for the Doroczy-Tsallis entropy for N particles (N oscillators)

$$S = \frac{kN}{q-1}\left(\sum P_i - \sum P_i^q\right), \quad P_i = \frac{N_i}{N}, \quad q = \text{non-extensive parameter.} \tag{1}$$

By varying Eq. (1) with respect to N_i and using the constraints

$$\sum N_i = const, \qquad \sum N_i \epsilon_i = const\,, \tag{2}$$

we find the following expression for N_i (μ/τ and $-1/\tau$ are Lagrange multipliers, where $\tau = kT$)

$$N_i = \frac{N}{q^{1/(q-1)}}\left(1 + \frac{\mu - \epsilon_i}{\tau}(q-1)\right)^{1/(q-1)}. \tag{3}$$

If we write $q - 1 = \alpha$ and expand $\mu = \mu_0 + \alpha\mu_1 + \alpha^2\mu_2$ (α = perturbation parameter, $q = 1$ for Boltzmann-Gibbs statistics), we find

$$N_i = N\mathrm{e}^{-1}\,\mathrm{e}^{(\mu_0-\epsilon_i)/\tau}\left[1 + \alpha\left(\frac{1}{2} + \frac{\mu_1}{\tau} - \frac{(\mu_0-\epsilon_i)^2}{2\tau^2}\right)\right]. \tag{4}$$

In Ref. [27] we found

$$\mathrm{e}^{\mu_0/\tau} = \frac{\mathrm{e}}{\sum \mathrm{e}^{-\epsilon_i/\tau}}\,, \qquad \mu_1 = \frac{\tau}{2} + \frac{1}{2\tau}\,\frac{\sum \mathrm{e}^{(\mu_0-\epsilon_i)/\tau}\,(\mu_0-\epsilon_i)^2}{\sum \mathrm{e}^{(\mu_0-\epsilon_i)/\tau}}\,. \tag{5}$$

If we consider the states of the harmonic oscillator $\epsilon_n = (n + \frac{1}{2})h\nu$, we find for the average energy of an oscillator

$$\langle\epsilon\rangle = \frac{\sum \epsilon_i N_i}{N}\,. \tag{6}$$

Using Eqs. (2.4), (2.5), (2.6) and $\epsilon_n = (n+\frac{1}{2})h\nu$, we found in Ref. [28] for $h\nu/\tau > 1$ (to order α)

$$\langle\epsilon\rangle = \frac{h\nu}{2} + \frac{h\nu}{\mathrm{e}^{h\nu/\tau} - 1} - \frac{\alpha}{8}\,\frac{(h\nu)^3}{\tau^2} + \frac{\alpha}{8\tau}(h\nu)^2\,. \tag{7}$$

If we now consider α to depend on frequency (such that $\alpha = 0$ for $h\nu/\tau < 1,\ \alpha = \alpha_0 + \alpha_1\nu^2$ for $h\nu/\tau > 1,\ \alpha_0 > \alpha_1$), we find that after subtracting off the vacuum term $h\nu/2$ that $\langle\epsilon\rangle = 0$ at a critical frequency ν_c, where

$$\frac{h\nu_c}{\mathrm{e}^{h\nu_c/\tau} - 1} - \frac{1}{8}(\alpha_0 + \alpha_1\nu_c^2)\,\frac{(h\nu_c)^3}{\tau^2} + \frac{1}{8\tau}(\alpha_0 + \alpha_1\nu_c^2)(h\nu_c)^2 = 0\,. \tag{8}$$

If we consider $\nu_c = \nu_0 + \bar{\epsilon}\nu_1$, where $\alpha_1 \to \alpha_1\bar{\epsilon}$ $(0 \le \bar{\epsilon} \le 1)$, we find the following solution to the transcendental equation in Eq. (8) (to first order in α_1)

$$\frac{h\nu_0}{\mathrm{e}^{h\nu_0/\tau} - 1} = \frac{1}{8}\alpha_0 \frac{(h\nu_0)^3}{\tau^2} - \frac{1}{8}\alpha_0 \frac{(h\nu_0)^2}{\tau}$$

or

$$h\nu_0 \mathrm{e}^{-h\nu_0/\tau} = \frac{1}{8}\alpha_0 \frac{(h\nu_0)^3}{\tau^2} - \frac{1}{8}\alpha_0 \frac{(h\nu_0)^2}{\tau}, \qquad \left(\frac{h\nu_0}{\tau} = \bar{C} >> 1\right), \tag{9}$$

and

$$\nu_1 \cong \frac{\alpha_1 \bar{C}^5}{8h^3}\,\tau^3\,, \tag{10}$$

(where we kept the dominant term in $\bar{\epsilon} \to 1$ and assumed $\alpha_0 (h\nu_0/\tau)^2 < 1$). Here $h\nu_0/\tau = \bar{C}$ solves Eq. (9). Thus

$$\nu_c = \frac{\bar{C}\tau}{h} + \bar{\alpha}\tau^3\,, \qquad \left(\bar{\alpha} = \frac{\alpha_1 \bar{C}^5}{8h^3}\right). \tag{11}$$

Thus for $\nu > \nu_c$, the spectrum cuts off and we assume $\langle\epsilon\rangle = 0$ for $\nu > \nu_c$ (after subtracting the vacuum term). Thus

$$\langle\epsilon\rangle = \frac{h\nu}{\mathrm{e}^{h\nu/\tau} - 1} \qquad \text{for} \qquad \frac{h\nu}{\tau} < 1\,, \tag{12}$$

and

$$\langle\epsilon\rangle = \frac{h\nu}{\mathrm{e}^{h\nu/\tau} - 1} - \frac{1}{8}(\alpha_0 + \alpha_1\nu^2)\,\frac{(h\nu)^3}{\tau^2} + \frac{1}{8}\left(\alpha_0 + \alpha_1\nu^2\right)\frac{(h\nu)^2}{\tau} \quad \text{for} \quad \frac{h\nu_c}{\tau} > \frac{h\nu}{\tau} > 1 \tag{13}$$

and $\langle\epsilon\rangle = 0$ for $\nu > \nu_c$. Actually $\langle\epsilon\rangle = 0$ for $\nu > \nu_c$ would require

$$\alpha(\nu) = \frac{h\nu}{(\mathrm{e}^{h\nu/\tau} - 1)\left(\dfrac{(h\nu)^3}{8\tau^2} - \dfrac{(h\nu)^2}{8\tau}\right)}\,, \tag{14}$$

which has the property $\alpha \to 0$ at $\nu \to \infty$. We will assume that the spectrum has a cut-off at ν_c and calculate the total energy. For the total energy per unit volume, we have from Eqs. (12) and (13)

$$\begin{aligned} U(\tau) \;=\;& \int\limits_0^{\tau/h} \frac{h\nu}{(\mathrm{e}^{h\nu/\tau} - 1)}\,\frac{8\pi\nu^2 \mathrm{d}\nu}{C^3} + \int\limits_{\tau/h}^{(\bar{C}\tau/h)+\bar{\alpha}\tau^3} \frac{h\nu}{(\mathrm{e}^{h\nu/\tau} - 1)}\,\frac{8\pi\nu^2 \mathrm{d}\nu}{C^3} \\ &+ \int\limits_{\tau/h}^{(\bar{C}\tau/h)+\bar{\alpha}\tau^3} \left(-\frac{(h\nu)^3}{8\tau^2} + \frac{(h\nu)^2}{8\tau}\right)(\alpha_0 + \alpha_1\nu^2)\,\frac{8\pi\nu^2 \mathrm{d}\nu}{C^3}\,, \end{aligned} \tag{15}$$

where C is the velocity of light. Changing integration variables to $x = h\nu/\tau$, $\nu = \tau x/h$, we find for Eq. (15) (letting $\bar{\epsilon} = 1$ in $\alpha = \alpha_0 + \epsilon\alpha_1\nu^2$ and $\nu_c = \nu_0 + \epsilon\nu_1$)

$$\begin{aligned} U(T) &= T^4 \frac{8\pi}{C^3} \frac{k^4}{h^3} \int\limits_0^{\bar{C}+\bar{\alpha}h\tau^2} \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} \\ &- T^4 \frac{\alpha_0 h^3}{8} \frac{8\pi}{C^3} \frac{k^4}{h^6} \int\limits_1^{\bar{C}+\bar{\alpha}h\tau^2} x^5 \,\mathrm{d}x - T^6 \frac{\alpha_1 h^3}{8} \frac{8\pi}{C^3} \frac{k^6}{h^8} \int\limits_1^{\bar{C}+\bar{\alpha}h\tau^2} x^7 \,\mathrm{d}x \\ &+ T^4 \frac{\alpha_0 h^2}{8} \frac{8\pi}{C^3} \frac{k^4}{h^5} \int\limits_1^{\bar{C}+\bar{\alpha}h\tau^2} x^4 \,\mathrm{d}x + T^6 \frac{\alpha_1 h^2}{8} \frac{8\pi}{C^3} \frac{k^6}{h^7} \int\limits_1^{\bar{C}+\bar{\alpha}h\tau^2} x^6 \,\mathrm{d}x \end{aligned} \tag{16}$$

(here α depends on α_1 to the first order).

If we expand Eq. (16) to first order in α_1, we obtain

$$U = \beta T^4 - \gamma T^6 \tag{17}$$

to first order in α_1.

For the pressure of the photon gas corresponding to Eq. (16), we have the relation

$$\left(\frac{\partial \bar{U}}{\partial V}\right)_T = T \left(\frac{\partial P}{\partial T}\right)_V - P\,. \tag{18}$$

Here $\bar{U} = U(T)V$, where $\bar{U}$ is the total energy in volume V and U is the energy per unit volume. From Eqs. (17) and (18), we have

$$U = T\,\frac{\mathrm{d}P}{\mathrm{d}T} - P\,,$$

or

$$\frac{\mathrm{d}P}{\mathrm{d}T} - \frac{P}{T} = \beta T^3 - \gamma T^5\,. \tag{19}$$

This equation can be written as $(\mathrm{d}/\mathrm{d}T)(P/T) = \beta T^2 - \gamma T^4$ and integration gives $P = \beta T^4/3 - \gamma T^6/5$. Thus, for the photon gas obeying Tsallis statistics with $\alpha = \alpha_0 + \alpha_1\nu^2$, for $\tau/h < \nu < \nu_c$, we have

$$\begin{aligned} U &= \beta T^4 - \gamma T^6 \\ P &= \beta T^4/3 - \gamma T^6/5\,. \end{aligned} \tag{20}$$

3. Conclusion

The above calculations suggest that both the energy density and the pressure receive corrections due to Tsallis-like modifications of the entropy of an ensemble of harmonic oscillators. In a previous note [33] using the Haldane approach to modified statistics (Ref. [8]), we calculated corrections to the equation of state for a photon gas when α varies just in the low frequency range (α = parameter in Haldane approach), just in the high frequency range, and for all ν. The corrections were all lower than the 4^{th} power of T. Also in Ref. [34], we calculated the corrections to U and P induced by the discrete spatial character of the space in which the photon propagates. One possible way to look for the corrections induced by Tsallis statistics in Eq. (20) is in the influence these modifications have on a cosmological evolution around the period of recombination. The equation of state in Eq. (20) will modify the temporal evolution of the cosmological scale factor leading to a modification of the temperature scale factor relation ($TR = C$) during the radiation era. This in turn will effect the red-shifting of the cosmic microwave background (CMB) from distance sources and lead to a distance inhomogeneity as well as the usual angular anisotropy of the CMB. In the conventional theory, there is no distance inhomogeneity because the radiation at any distance is red-shifted down to the present wavelength of the CMB (enforced by $TR = C$). Thus a slight variation of a CMB energy density versus distance would signal the presence of Tsallis modified statistics for photons. In the early universe, both anomalies in photon statistics and anomalies due to quantum-gravity-induced discrete spatial effects would have to be taken into account in the calculation of the equation of state of radiation that drives the cosmological expansion. Recent studies [35,36] in non-commutative geometry have suggested discrete spatial effects and modifications in particle statistics that could very well modify the equation of state of photons driving the early universe expansion. Lastly, if the calculation of the modified energy density leads to a modified pressure that turned negative, it might serve to suggest that inflation [37] can result from a Tsallis like description of the photon gas without the contrived use of Higgs potentials based on the uncertainties of C.U.T. theories [38].

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JEDNADŽBA STANJA ZA FOTONE PREMA TSALLISOVOJ STATISTICI

Pretpostavkom da raspodjela stanja harmoničkog oscilatora na visokim frekvencijama slijedi multifraktalnu raspodjelu, pokazujemo da spektralna raspodjela energije za fotone koji slijede izmijenjenu statistiku ima anomalna svojstva, te se stoga promijeni i jednadžba stanja fotonskog plina.