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INHERENTLY RELATIVISTIC QUANTUM THEORY
Part I. THE ALGEBRA OF OBSERVABLES

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Dedicated to Professor Kseno Ilakovac on the occasion of his 70th birthday

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The present article is the first in a program that aims at generalizing quantum mechanics by keeping its structure essentially intact, but constructing the Hilbert space over a new number system much richer than the field of complex numbers. We call this number system "the Quantionic Algebra". It is eight dimensional like the algebra of octonions, but, unlike the latter, it is associative. It is not a division algebra, but "almost" one (in a sense that will be evident when we come to it). It enjoys the minimum of properties needed to construct a Hilbert space that admits quantum-mechanical interpretations (like transition probabilities), and, moreover, it contains the local Minkowski structure of space-time. Hence, a quantum theory built over the quantions is inherently relativistic. The algebra of quantions has been discovered in two steps. The first is a careful analysis of the abstract structure of quantum mechanics (the first part of the present work), the second is the classification of all concrete realizations of this abstract structure (several additional articles). The classification shows that there are only two realizations. One is standard quantum mechanics, the other its inherently relativistic generalization. The present article develops the abstract algebra of observables.

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1. Introduction

Following the discovery of non-relativistic quantum mechanics, much effort has been invested into the search for an encompassing supra-theory that would be simultaneously quantum mechanical and general relativistic, but a structural unification of space-time and quantum theory is still elusive. As for the unifications provided by quantum field theories, they are not structural in the mathematical sense. Relying on phenomenology, they adapt quantum theory to particular fields — as mirrored in expressions like "quantization of ...(whatever field)". By contrast, the objective of structural unification is to discover, *if it exists,* a generalization of quantum mechanics that would be inherently relativistic. Thus, it is quantum theory itself that would be generalized — "generalization" being understood in the following sense: *A generalization of an axiomatic theory is a new axiomatic theory which implies the initial one in some carefully defined limit.* Such structural generalizations have already taken place in physics, as illustrated in Fig. 1. The present work attempts a systematic approach to the last step, "Generalized Q.M.". We note that the generalization problem, so understood, is purely mathematical. It does not aim at incorporating more phenomenology into a theoretical model, but at discovering new mathematical structures recognizable as quantum mechanics in all essential aspects. If solved, the new theory may or may not be physically meaningful. To be physical, it must also admit experimentally testable physical interpretations and agree with observations over a wider range of phenomena than the original theory.

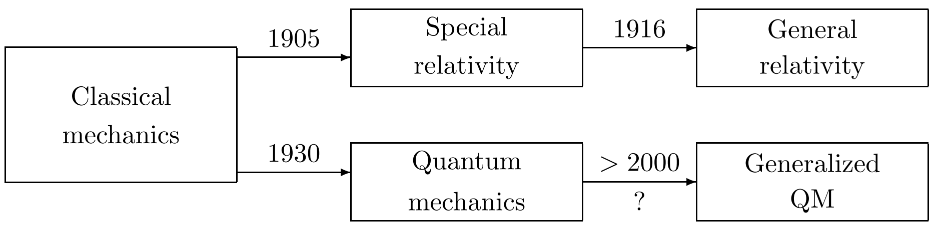

Figure 1. The Generalizations of classical mechanics.

As stated above, we aim at generalizing quantum mechanics while preserving its essential aspects, but what is considered essential is often a matter of physical intuition and personal philosophy. In a systematic attempt to eliminate any preconceived ideas one might have about what physics "should be", part of the present work (the present paper in totality) is dedicated to the structural analysis of standard quantum mechanics. Having extracted this mathematical structure, referred to as "quantal algebra", what follows is deductive work, which begins with classification, i.e., the identification of all concrete realizations of this abstract algebra. Clearly, not much would be achieved if the number of realizations were infinite, or even large, forcing us to search in a large pool of candidates for those that might be physically relevant — but it happens that there is only one non-standard realization, and, unexpectedly, it is relativistic. Thus, our initial objective of constructing a structural unification of quantum mechanics and relativity is achieved at a deeper level than expected. *The abstract structure of quantum mechanics implies the existence of space-time with Minkowski structure*, which justifies the terminology "inherently relativistic".

The present multi-part work's key contributions to the unification problem (developed much later in the work) is the discovery[1] of a new number system, referred to as "Quantionic Algebra", with the following properties:

- It is the only generalization of the field of complex numbers which allows the construction of a Hilbert space with all properties needed for quantum mechanical interpretation.
- It is covariant with respect to the Loentz group $SO(1,3)$, even though this property is not axiomatically imposed.
- It can be expressed in a formalism well adapted to differential geometry in curved manifolds, which may be relevant to the vexing problem of quantum gravitation.

The first point calls for a brief comment, as it may seem puzzling to the reader acquainted with Hurwitz's theorem. Among the properties needed for quantum mechanical interpretation is a positive definite norm, but according to the theorem in question, the only number systems equipped with a positive definite norm are the four division algebras, i.e., the real numbers, the complex numbers, the quaternions, and the octonions — leaving no room for a new normed number system. The quantionic algebra does not challenge this theorem, but circumvents it in a rather subtle manner. As we shall see in a much later part of the present work, the vectors in a properly defined quantionic Hilbert space have a positive definite norm even though their components don't — and it is the vectors, not the components, that must have a norm if the probabilistic interpretation is to be preserved.

This may also be the place for a comment concerning generalizations of quantum mechanics. As strongly argued by Steven Weinberg [1], it is established common wisdom, supported also by some theorems, that the Hilbert space structure of quantum mechanics is so rigid that it cannot be modified, discouraging attempts at generalization. Our results, derived in later parts of the present work, only confirm this observation, as our generalization of quantum mechanics preserves the structure of Hilbert space and its physical interpretations. What is different is *the realization of Hilbert space*, which is no longer over the field of complex numbers, but over a new number system. What may be unexpected, is the existence of a number system which would modify quantum mechanics without destroying its essential Hilbert space structure.

Returning from these digressions, we would expect a generalization of quantum mechanics to exist due to the apparent incompleteness of the interconnections between the basic theories of physics, as illustrated in Fig. 2.

Arrows indicate specializations (transitions to special or limiting cases), while solid lines connect the various theories to the space-time groups they are compatible with. As in Fig. 1, classical mechanics appears as a limit of two theories — relativity and quantum mechanics. It shares with the latter the same space-time group, while

[1] Whether it is 'discovery" or "construction" depends on one's personal philosophy of mathematics. The author's view favors "discovery", as it is impossible to construct a mathematical structure that does not exist in principle — and if it does, its "construction" is pre-determined.

its relation to relativity takes place by way of a contraction of the Poincaré group to the Galileian group. The physical theories on the two sides of the graph have been technically complete within their ranges of validity for most of the Twentieth Century, but their logical disconnectedness is unsatisfactory. It suggests that a "missing link" remains to be discovered. The figure calls for completion to a lattice structure, i.e., to a graph that would contain not only a minimally structured theory at the bottom (classical mechanics), but also a maximally structured one at the top. Both sides of the graph would then be derived from such a supra-theory, which, ideally, would also admit gravitation. The new relationships are tentatively illustrated in Fig. 3.

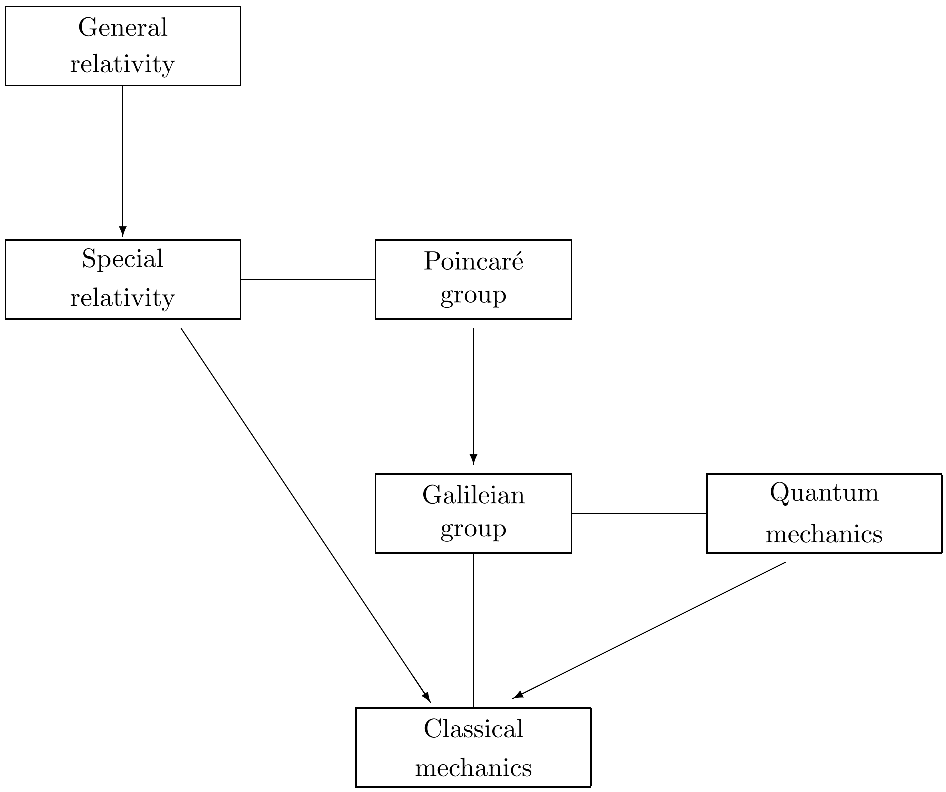

Figure 2. The Current structure of physics.

Our objective being to generalize quantum theory, let us briefly review the heuristics of our approach. The problem may be stated as follows:

Given two concrete theories (referred to as **prototype theories** *), construct a new abstract mathematical theory that encompasses both, and then find all other concrete theories of the same type, i.e., all realizations of the new abstract structure.*

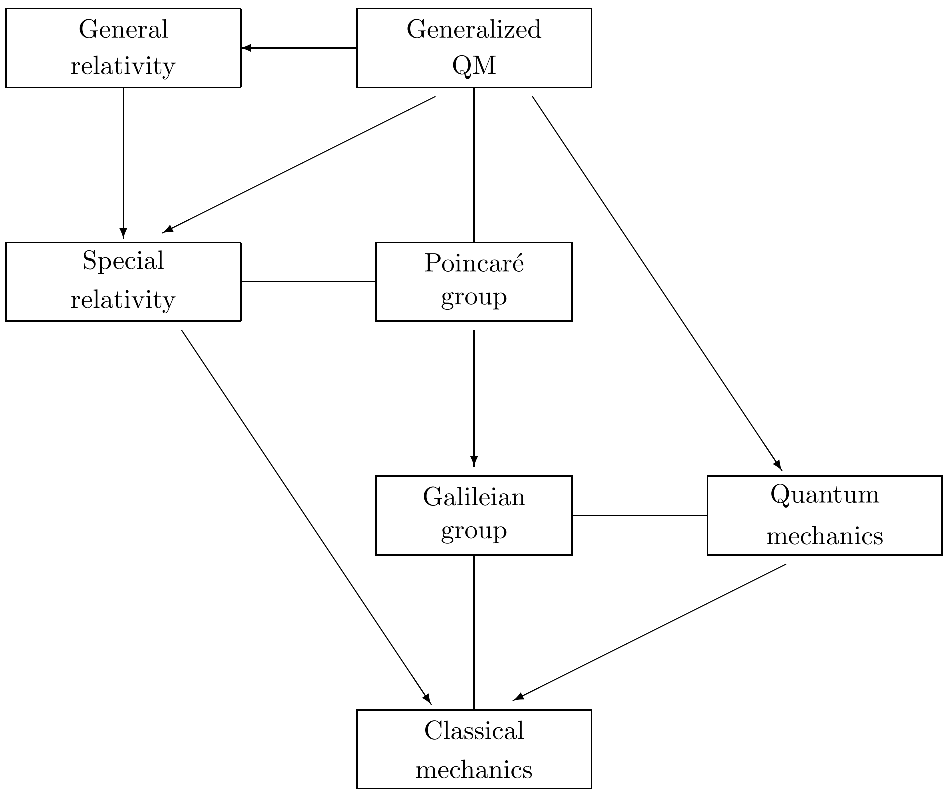

Figure 3. A new hypothetical model of physics.

A three-step approach to this task is illustrated in Fig. 4. The first step, *concept extraction,* consists in identifying the mathematical concepts common to the prototype theories and selecting the ones that are to be built into the foundations of the abstract theory. The advantage of starting with two prototypes, if available, is obvious, as this greatly reduces the set of candidate concepts — whatever is not common to both is considered inessential. The second step consists in *structuring* these basic concepts into an abstract mathematical object. If this can be done, the last step is the *classification of concrete realizations.* A concrete realization of an abstract mathematical structure is a mathematical object which exhibits all the properties of the abstract structure, but is defined in terms of a pre-existing theory that allows practical computations. Obviously, the prototype theories themselves are realizations of the abstract theory, but not necessarily the only ones — and it is among the new ones that one might expect to find a generalization of quantum theory.

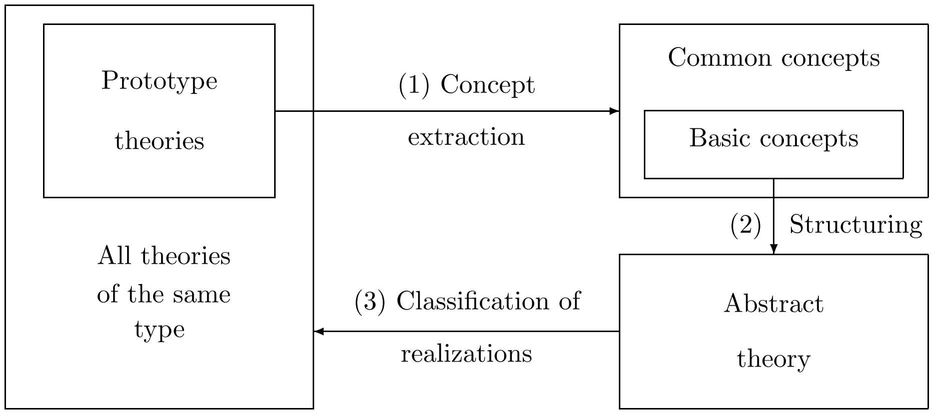

Figure 4. The generalization procedures

Following the generalization procedures outlined above, our first task is to select the prototype theories. One of them is quantum mechanics by definition. One more is needed to limit and guide the concept selection process. Our ultimate objective, which is the unification of quantum mechanics and relativity, might suggests that the second theory should be special relativity, but it cannot be, as quantum mechanics and relativity refer to very different aspects of physics. The only choice is classical mechanics. Though a limiting case of quantum mechanics, classical mechanics is not a subset of the latter. The two theories being axiomatically unrelated, they can serve as prototypes without fear of redundancy. Concerning the choice of formulations, Heisenberg's picture and Hamilton's canonical formalism suggest themselves, as they exhibit the most pronounced structural analogies. As for the idea of generalizing the complex numbers, it was first suggested by a result obtained with A. Petersen [2] in the structural formulation of Bohr's correspondence principle, where the transition from quantum to classical mechanics may be viewed as a transition from the field of complex numbers to a nilpotent algebra[2]. A suggestive

[2]This takes place in the phase space formulation of quantum mechanics [2–3]. In this formulation, we denote the Poisson bracket as a formal product, ∇, i.e., $f\nabla g \stackrel{\text{def}}{=} \{f,g\}$, so that its powers, ∇^n, are conveniently defined as bilinear differential operators of order $2n$. The algebraic structure of phase space quantum mechanics is then contained in a complex product of phase space functions, $f\, e^{\mathrm{i}h\nabla} g$. This is the only associative product which can be defined in phase space in addition to the ordinary product of functions, fg. The correspondence principle may then be viewed either as a numerical limit, or as a structural deformation. The first view is applicable if the action in the physical system is large enough for Planck's constant to be negligible. The complex product then splits into the two real products that characterize the structure of classical mechanics, namely fg and $\{f,g\}$. This is the historical view of correspondence, but, according to Aage Petersen (oral communication) Bohr often expressed the opinion that there must be something deeper to this principle. The second view confirms this opinion, as we can also retrieve the algebra of classical mechanics by a structural deformation of the product $f\, e^{\mathrm{i}h\nabla} g$. Indeed, we may take $h = 1$, and consider a formal product $f\, e^{J\nabla} g$. For $J = \mathrm{i}$, we have quantum mechanics, for J nilpotent, i.e.,

question is now inescapable: If the transition from the field of complex numbers to a nilpotent algebra effects the structural transition from quantum to classical mechanics, might there not exist a structurally much richer algebra, $\mathbb{D}$, such that the transition $\mathbb{C} \to \mathbb{D}$ would lead in the opposite direction, i.e., from quantum mechanics to, one might hope, a "covariant quantum theory"? It is to this remark involving the nilpotent algebra that the present work owes its origin, even though the idea of generalizing quantum mechanics by substituting a richer structure for the field of the complex numbers is almost as old as quantum mechanics itself, the large body of research in this area being well reviewed in several books [4–7]. Our approach differs from all these previous works in that it does not attempt to generalize quantum mechanics by forcing it into mathematical structures already known to mathematics. Instead, we assume from the outset that a number system which would generalize the complex numbers in a manner fully consistent with quantum mechanics either does not exist, or, if it does, is not known to mathematics — for, if it were known, its relevance to physics would not have been left unnoticed for over half a century. Guessing being out of the question, the path we take to discovering this hypothetical structure is a careful abstract analysis of quantum mechanics followed by its step-by-step reconstruction. The expectation being that this process will bring to light some hidden assumptions whose elimination might open the door to generalization. As our approach takes a new direction starting from first principles, it is independent of any previous research on unification.

The present paper is dedicated to the construction of the abstract generalization of quantum mechanics as an algebra of observables. In subsequent parts of the work we shall develop the abstract theory of states, classify all concrete realizations of the abstract theory, prove the existence and uniqueness of a generalization of the field of complex numbers (referred to as the algebra of quantions), study the properties of this new number system, develop a generalization of Hilbert space by substituting quantions for complex numbers, etc..

2. *The process of generalization*

Quantum mechanics is based on a system of postulates which fall under three headings: *Observables* — represented by Hermitian operators in a Hilbert space. *States* — represented by rays in the same Hilbert space. *Probabilities* — computed as norms of scalar products in Hilbert space.

Our ultimate goal is to preserve these postulates in the new theory while mod-

$J^2 = 0$, we retrieve classical mechanics. The relationship between the two interpretations of J is more transparent in a two-dimensional real representation, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for the imaginary element, and $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for the nilpotent element. This correspondence principle based on J does not model microscopic physics, but sheds light on the theory by showing that the algebra of both mechanics can be expressed by a single associative product, $f\, e^{J\nabla} g$. This product is based on the number system $\mathbb{R} \oplus J\mathbb{R}$, which is the field $\mathbb{C}$ of complex numbers in the quantum case, and on a nilpotent algebra in the classical case.

ifying only the realization of Hilbert space by substituting a new number system for the field of complex numbers. To discover this hypothetical number system, we first rewrite as much as possible of quantum mechanics in a form that involves no complex numbers, and then re-introduce the role they play in the theory by way of an inequivalent construction. To this end, we first note that while the components of rays are complex numbers by definition, Hermitian operators are real objects. If, as an intermediate step, the states were also formulated as real objects, we would be much closer to our goal of eliminating the complex numbers from the theory. The last step would be a process of abstraction by which the internal representation of objects whose matrix components are still complex numbers would be ignored, and only relationships considered essential.

A formulation of quantum mechanics in which states are also observables, hence real objects, does in fact exist. It is in terms of density matrices. Its postulates make no explicit reference to Hilbert space, and, in addition, are analogous to those of canonical classical mechanics. Grouped as above and comparatively listed, they are:

1) (B1): *Observables.*

QM: Represented by Hermitian operators.

CM: Represented by real functions over phase space.

1) (B2): *States.*

QM: Represented by idempotent Hermitian operators of unit trace (density matrices).

CM: Represented by probability densities over phase space (the Dirac function in the non-statistical theory).

1) (B3): *Transition probabilities.*

QM: Computed as traces of products of observables and states.

CM: Computed as integrals of products of observables and states.

Having selected the prototype theories of mechanics in Hamilton's and Heisenberg's forms, the first step, as stated in the introduction, is to extract the set of common concepts and properties. We shall see that this set is categorical, i.e., neither too small (insufficient) nor too large (inconsistent), but "just right" to enable us to construct a unique abstract formulation of both mechanics. This is not surprising if one notes that the realizations of these theories in terms of Hermitian matrices on the one hand, and of real functions over phase space on the other, have many properties in common even though they are mathematically very far apart. It is virtually inconceivable that structures formally so distant from each other could be so similar by some inconsequential coincidence.

2.1. Concept extraction

The following is a compilation of the properties we can readily identify as common to real functions over phase space and to Hermitian matrices.

- *Linearity.* The set of phase space functions and the set of Hermitian matrices, i.e., the observables in both mechanics, form linear spaces over the field of real numbers.

- *Reality.* In both theories, we distinguish three ways in which reality manifests itself: (1) The reality of the coefficients over which the linear space of observables is built. (2) The reality of the values associated to the observables (e.g., the eigenvalues of Hermitian operators are real). (3) The "intrinsic" reality of the observables themselves (e.g., Hermitian matrices are self-adjoint). These manifestations of reality coalesce into a single one in classical mechanics, and are mutually related in quantum mechanics. As we don't know whether they will be related or independent in a more general theory, we initially view them as independent.

- *Existence of a product.* Phase space functions may be mutually multiplied to yield new phase space function, while Hermitian matrices yield new Hermitian matrices by the symmetric product. Hence, in both theories, the linear spaces of observables are equipped with a symmetric (i.e., commutative) product.

- *Automorphisms.* The canonical and unitary transformations are automorphisms of the products mentioned above.

- *Reversibility.* In "pure" (as opposed to statistical) classical and quantum mechanics there are no irreversible automorphisms (no entropy). Thus, the automorphisms form a group.

- *Inner derivations.* Derivations define the infinitesimal automorphisms. In both theories, all derivations are inner, being defined by the observables themselves — either by way of the Poisson bracket, or by the commutator. This property is referred to as "the equivalence of observables and generators".

- *Composition.* Instances of both theories compose without restrictions, allowing physical systems to interact. Under composition, the degrees of freedom are additive, and the canonical and unitary structures are preserved.

The large number and generality of these concepts strongly suggests that it should be possible to formulate them within an abstract algebraic structure free of reference to Hilbert space or phase space — which is the next step, referred to as "structuring".

2.2. Structuring

The formulation of any abstract theory begins with the selection of undefined concepts, or primitives — objects considered indecomposable into more elementary concepts. We take as primitive only the concept of **observable**, leaving the door open to additional ones if necessary. Abstract observables, denoted by small Latin letters, $f, g, h, ...$, are internally unstructured by definition (they are neither phase space functions not operators), and the set of observables, denoted by $\mathcal{O}$, is also initially unstructured. By going through the list of previously extracted common concepts, we shall add structure to $\mathcal{O}$ step by step in the next subsections

2.3. Linearity

In both classical and quantum mechanics, the observables form a real linear space: linear combinations of real functions are real functions, and linear combinations of Hermitian operators are Hermitian, if and only if, in both cases, the coefficients are real. Hence, real linearity is the first property we take over as essential:

$$(f, g \in \mathcal{O}) \,\&\, (\lambda, \mu \in \mathbb{R}) \Rightarrow (h = \lambda f + \mu g \in \mathcal{O}) \,.$$

2.4. The symmetric product

The observables of classical mechanics, also called dynamical variables, are the C^∞ functions over phase space. This class of functions is stable under the ordinary product, which is commutative and associative. In quantum mechanics, the associative product AB of Hermitian matrices is not Hermitian, and, consequently, is not a candidate for abstraction, but the Jordan product $\frac{1}{2}(AB + BA)$ is Hermitian and symmetric (though not associative). Hence, what is common to both mechanics is a symmetric product — which makes it an essential concept.

We thus postulate the existence in the abstract theory of a symmetric product, which we denote by σ. For $f, g \in \mathcal{O}$ we write $h = f\sigma g \in \mathcal{O}$. As stated above, this product satisfies the identity

$$f\sigma g = g\sigma f, \tag{1}$$

but no assumptions are made concerning its associativity. The associator, defined as the trilinear map

$$[f, g, h] \overset{\text{def}}{=} (f\sigma g)\,\sigma h - f\sigma\,(g\sigma h)\,, \tag{2}$$

measures the extent to which associativity is violated. As it vanishes in classical, but not in quantum mechanics, we introduce the following definition:

Definition 1 ***The classical criterion***: *The identity* $[f, g, h] = 0$ *for all* $f, g, h \in \mathcal{O}$ *characterizes abstract classical mechanics.*

We now require that the operations introduced so far (real linearity and the commutative σ-product) be mutually compatible, meaning that the order in which they are performed is to be irrelevant. This implies the standard distribution laws of algebra:

$$(f+g)\,\sigma h = f\sigma h + g\sigma h, \tag{3}$$

$$(\lambda f)\,\sigma g = \lambda\,(f\sigma g)\,, \tag{4}$$

$$(\lambda+\mu)\,f = \lambda f + \mu f. \tag{5}$$

Hence, the abstract structure $\{\mathcal{O},\sigma\}$ is a **real symmetric algebra**.

Another property common to both mechanics is the existence of a unit (the real unit and the unit matrix respectively). Hence, we postulate the existence of an observable, $e \in \mathcal{O}$, such that

$$e\,\sigma f = f \tag{6}$$

for all $f \in \mathcal{O}$. We call e **the abstract unit**. Thus, the algebra of observables, $\{\mathcal{O},\sigma,e\}$, is an algebra with a unit.

2.5. Reality of spectra

For every element f of an algebra with a unit, the **spectrum**, $Sp\,(f)$, is defined as the set of constants λ for which the element $(f-\lambda\,e)$ has no inverse. Clearly, for $(f-\lambda\,e)$ to be an element of the algebra, the constant λ must be an element of the field over which the underlying linear space of the algebra is constructed. In our case, the algebra is $\{\mathcal{O},\sigma,e\}$ and the field is $\mathbb{R}$. Hence, the spectrum $Sp\,(f)$ of any observable $f \in \mathcal{O}$ is the set of real numbers λ for which no unique observable $g \in \mathcal{O}$ exists such that the relation

$$(f-\lambda\,e)\,\sigma g = e$$

would be satisfied. This is guaranteed in classical and quantum mechanics by the realizations of observables as real functions or as Hermitian operators. Since we are ultimately interested in realizations of the algebra of observables by concrete objects other than standard Hermitian matrices, it is conceivable that the reality of spectra might not be an intrinsic property of these objects. Hence, we are *postulating the reality of the spectrum.*

2.6. Reality of observables

This additional concept of reality is not to be confused with the reality of the field, postulated earlier, even though it implies it. In classical mechanics, it corresponds to the observables being real functions, $\bar{f}(x,p) = f(x,p)$; in quantum mechanics, to their being self-adjoint operators, $F^{\dagger} = F$. Thus, it expresses an *internal* property of the observables (how they are individually constructed), while the reality of the field expresses an *external* property (how they combine to form new observables).

To formulate the concept of reality for the observables in the abstract algebra $\{\mathcal{O}, \sigma, \alpha\}$, we may use the general definition of conjugate elements, adapting it to this algebra.

To this end, let $\{\mathcal{O}', \sigma, e\}$ denote some extension of the algebra $\{\mathcal{O}, \sigma, e\}$, where $\mathcal{O}'$ generalizes the concept of complexification. For a set of elements $f_1, f_2, \ldots, f_n \in \mathcal{O}'$, we say that they are mutually conjugate if

$$f_1 + f_2 + \cdots + f_n \in \mathcal{O}, \tag{7}$$

$$(f_1 \sigma f_2)\, \sigma \cdots \sigma f_n \in \mathcal{O}, \tag{8}$$

while $f_i \notin \mathcal{O}$ for all $i = 1$ to n. This concept is a direct generalization of conjugation in the field of complex numbers. Indeed, if we take $\mathcal{O} = \mathbb{R}$ and $\mathcal{O}' = \mathbb{C}$, then the conditions $f_1 + f_2 \in \mathbb{R}$, $f_1 f_2 \in \mathbb{R}$, and $f_1, f_2 \notin \mathbb{R}$, imply $f_1, f_2 \in \mathbb{C}$. In particular, it follows that $f_2 = f_1^*$.

We may now formulate the abstract definition of reality for abstract observables as the requirement that all relations of the type 7 and 8 should have solutions in the space $\mathcal{O}$ itself. Like the reality of the spectrum, this concept of reality is automatically satisfied in classical and quantum mechanics, but might not be in a more general theory — which is why we postulate it.

2.7. Automorphisms

The invariance groups of classical and quantum mechanics are, respectively, the groups of canonical and of unitary transformations. Both are the groups of automorphisms of the products of observables. But this property need not be introduced into the abstract theory by an explicit conceptualization process, as it is mathematically implied. Indeed, given an abstract algebra $\{\mathcal{O}, \sigma, e\}$, its group of automorphisms is uniquely defined if $\mathcal{O}$ is finite-dimensional, which is all we need initially. This group is the set of all mappings $\mathbf{T} : \mathcal{O} \to \mathcal{O}$ which preserve linearity and the product σ, i.e., they are the mappings which commute with these operations. Formally, the commutativity conditions reads

$$\mathbf{T}\,(\lambda f + \mu g) \;=\; \lambda \mathbf{T}\,(f) + \mu \mathbf{T}\,(g) \tag{9}$$

$$\mathbf{T}\,(f \sigma g) \;=\; \mathbf{T}\,(f)\, \sigma \mathbf{T}\,(g) \tag{10}$$

for all $\lambda, \mu \in \mathbb{R}$, and all $f, g \in \mathcal{O}$. From the last relation follows the invariance of the unit,

$$\mathbf{T}\,(e) = e. \tag{11}$$

Hence, there is no need to rely on the canonical and unitary groups as physical prototypes, since the group of abstract automorphisms is distinguished as an implied structure once the algebra $\{\mathcal{O}, \sigma, e\}$ has been postulated to exist.

An infinitesimal automorphism is of the form

$$\mathbf{T} = I + \varepsilon D_T. \tag{12}$$

Substitution of this expression into relation (10) yields the Leibnitz derivation rule for the linear operator D_T:

$$D_T\,(f\sigma g) = (D_T f)\,\sigma g + f\sigma\,(D_T g)\,. \tag{13}$$

From relations (11) and (12) follows

$$D_T e = 0. \tag{14}$$

The commutator of derivation operators,

$$[D_T, D_S] = D_T D_S - D_S D_T, \tag{15}$$

is a derivation operator, i.e.,

$$[D_T, D_S]\,(f\sigma g) = ([D_T, D_S]\,f)\,\sigma g + f\sigma\,([D_T, D_S]\,g)\,,$$

and satisfies the Jacobi identity

$$[D_T, [D_S, D_R]] + [D_R, [D_T, D_S]] + [D_S, [D_R, D_T]] = 0. \tag{16}$$

It is thus a Lie product.

The operators D_X are the abstract counterparts of the canonical derivation based on the Poisson bracket, $\{X, \}$, and of the unitary derivation based on the commutator, $\frac{1}{\mathrm{i}}\,[X,]$. We disregard for the time being the fact that these operators are tied to an additional characteristic property of both mechanics (the equivalence of observables and generators), which will be discussed separately. Though abstract, the operators D_X are not structurally irreducible. They contain three structures which can be dissociated: (a) they form a linear space, (b) they compose by way of a product (the commutator), and (c) they act as linear operators on the space of observables. We shall discuss each in turn.

2.8. The linear space of generators

We shall now dissociate the three structures mentioned above and re-introduce them explicitly.

Linearity. Extracting linearity first, we introduce an abstract real linear space, $\mathcal{L}$, isomorphic with the linear space of differential operators D_X. We denote the elements of $\mathcal{L}$ by capital Latin letters, $F, G, ... \in \mathcal{L}$. Like the abstract observables, they are to be viewed just as points in a linear space, devoid of any internal structure. Specifically, they are not operators. We refer to them as (abstract) **generators**.

The operator structure. We explicitly restore the operator property to the abstract generators by introducing a hybrid product which we denote by γ. This product is a bilinear function from generator-observable pairs into observables:

$$\gamma : \mathcal{L} \times \mathcal{O} \rightarrow \mathcal{O}.$$

With $T \in \mathcal{L}$ and $f \in \mathcal{O}$, we write it as a mixed product, $T\gamma f \in \mathcal{O}$. Dropping the observable on which it acts, the relationship between a derivation operator D_F and its abstract representant F is

$$D_F \leftrightarrow F\gamma. \tag{17}$$

Substitution of this expression into the Leibniz rule (13) yields the distribution law for γ with respect to σ :

$$H\gamma\,(f\sigma g) = (H\gamma f)\,\sigma g + f\sigma\,(H\gamma g)\,, \tag{18}$$

while relations (14) and (17) yield

$$F\gamma e = 0\,. \tag{19}$$

The Lie structure. Since the commutator $[D_F, D_G]$ of derivation operators is a derivation operator and a Lie product, the equivalent form $[F\gamma, G\gamma]$ has the same properties — which implies that the linear space $\mathcal{L}$ is equipped with a Lie product. We denote this new product by ω. Its defining identity is

$$(F\omega G)\,\gamma = [F\gamma, G\gamma] \tag{20}$$

Hence, the linear space $\mathcal{L}$ supports an abstract Lie algebra $\{\mathcal{L}, \omega\}$. In other words, the product ω satisfies the antisymmetry and Jacobi identities

$$\begin{aligned} F\omega G + G\omega F &= 0 && (21)\\ (F\omega G)\,\omega H + (H\omega F)\,\omega G + (G\omega H)\,\omega F &= 0 && (22)\end{aligned}$$

in addition to distribution rules of the form (3) to (5), in which the product σ is to be replaced by the product ω.

Collecting all abstract objects introduced so far, we denote the composite algebraic structure by $\mathcal{Q}$:

$$\mathcal{Q} = \{\mathcal{O}, \mathcal{L}, \sigma, \omega, \gamma, e\}\,.$$

As this is an intermediate object, soon to be further specialized, there is no need to name it.

2.9. Composability

Another property implicit in the realizations of the two mechanics allows the construction of larger systems from smaller ones — the concept of system "size" being defined in classical mechanics as the number of degrees of freedom (half the dimension of phase space). We refer to the property in question as **composability**. To discuss the idea, we consider two physical systems in both mechanics, and determine from these examples which mathematical properties are needed to support interactions.

In classical mechanics, a system is represented by its Hamiltonian and various symmetry groups, but we don't need such a detailed specification to discuss composability. All we need is the fact that Hamiltonians are dynamical variables. Hence, the mathematics needed to represent a physical system is its phase space and the linear space of observables, i.e., of real C^{∞} functions over it.

Let us now consider two arbitrary mechanical systems, each with its phase space and linear space of observables. For these systems to be able to interact, there must exist a composite phase space and a linear space of composite observables. This composite structure is obtained by combining the two phase spaces into a larger one by a direct sum — which is technically trivial because the symplectic structure of phase space allows unrestricted addition of degrees of freedom. The new observables are then C^{∞} functions over the larger phase space.

To reformulate this rule in terms of observables alone, without making explicit reference to phase space (as the latter is not a common concept), we note that a composite observable can be viewed as a (possibly infinite) sum of products of component observables. Thus, given a C^{∞} composite function $h\left(x_1, p_1, x_2, p_2\right)$, one can find two sequences of C^{∞} functions, $f_k\left(x_1, p_1\right)$, $g_k\left(x_2, p_2\right)$, such that

$$h\left(x_1, p_1, x_2, p_2\right) = \sum_k f_k\left(x_1, p_1\right) g_k\left(x_2, p_2\right).$$

Hence, disregarding the technically fine points of convergence — which are irrelevant since the example is used only for heuristic purposes — we make note of the fact that the underlying linear spaces of observables compose by way of the tensor product.

A similar conclusion holds in quantum mechanics, where each system is represented by its Hilbert space and by the associated linear space of Hermitian operators — the Hamiltonian being irrelevant to this discussion as well. Disregarding statistics at this point (i.e., considering only non-identical systems), the composite structure is obtained by taking the tensor product of the Hilbert spaces, which then implies that the new space of observables is the tensor product of the component spaces of observables. Just as the tensor product of classical observables is compatible with the symplectic structure of phase space, so is the tensor product of Hermitian matrices compatible with Hermiticity. Indeed, if we take an arbitrary n-dimensional Hermitian matrix and multiply each of its components by an arbitrary m-dimensional Hermitian matrix, the resulting nm-dimensional matrix is also Hermitian.

Hence, any two instances of classical mechanics can compose to yield a new one, and the same is true for any two instances of quantum mechanics. These theories do not cross-compose, however, which is consistent with the non-existence of hybrid classical-quantum theories.

Initially disregarding the insight that composition takes place by way of tensor products, all we need to extract from these observations is the conclusion that composition exists. We may also note that it takes place within composition classes — classical and quantum mechanics being two such classes. The reason it is not necessary to extract from the examples just considered the postulate that abstract composition is based on the tensor product is that it cannot be otherwise, since the linearity of the spaces $\mathcal{O}$ and $\mathcal{L}$ is to be preserved by composition, as is the bilinearity of the products σ, ω and γ.

On the basis of these considerations, we expect all objects $\mathcal{Q}$ to fall into **composition classes** (the classical and quantum theories being two prototype classes). We denote the composition operation by "$\bigcirc$". Thus, if $\{\mathfrak{T}, \bigcirc\}$ is a composition class, then, for any two objects $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathfrak{T}$, we postulate the existence of a third object, $\mathcal{Q}_{12} \in \mathfrak{T}$, which is their composite:

$$\mathcal{Q}_{12} = \mathcal{Q}_1 \bigcirc \mathcal{Q}_2. \tag{23}$$

Writing

$$\begin{aligned} \mathcal{Q}_1 &= \{\mathcal{O}_1, \mathcal{L}_1, \sigma_1, \omega_1, \gamma_1\}, && (24) \\ \mathcal{Q}_2 &= \{\mathcal{O}_2, \mathcal{L}_2, \sigma_2, \omega_2, \gamma_2\}, && (25) \\ \mathcal{Q}_{12} &= \{\mathcal{O}_{12}, \mathcal{L}_{12}, \sigma_{12}, \omega_{12}, \gamma_{12}\}, && (26) \end{aligned}$$

we see that the symbol $\bigcirc$ represents a set of five still unknown mappings, which are to define:

- the two linear spaces $\mathcal{O}_{12}$ and $\mathcal{L}_{12}$ in term of the spaces $\mathcal{O}_1$, $\mathcal{L}_1$, $\mathcal{O}_2$, $\mathcal{L}_2$, and
- the three products σ_{12}, ω_{12}, γ_{12} as functions of the six products σ_1, ω_1, γ_1; σ_2, ω_2, γ_2.

2.10. Grading

By integration, relations (12) and (17) yield finite one-parametric motions of the type

$$\exp\left(\tau T\gamma\right) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left(T\gamma\right)^k .$$

The parameter τ has no intrinsic physical meaning, so that its scale and orientation are arbitrary. Considering the latter, we note that the parameter reversal $\tau \to -\tau$ is equivalent to the generator reversal $T \to -T$. This is consistent with the reversibility property of the automorphisms. Generalizing to a simultaneous reversal of all generators in a object $\mathcal{Q}$, we define the reversal operator $\mathbf{R}$ in $\mathcal{Q}$ by the relations

$$\mathbf{R}|_{\mathcal{O}} = I, \tag{27}$$

$$\mathbf{R}|_{\mathcal{L}} = -I, \tag{28}$$

which mean that every observable remains invariant and every generator changes sign. For this mapping to be a symmetry of the theory, it must be compatible with the composition operation $\bigcirc$, which implies that it is to be performed simultaneously on all objects $\mathcal{Q}$ in a composition class. It thus follows that $\mathbf{R}$ is a class operator, in the sense that it leaves composition classes invariant.

An elegant way of formulating this conclusion is with the concept of "grading". In general, a linear space is said to be graded if it is to be viewed as a direct sum of linear subspaces, and if each of these subspaces is associated to an element of some Abelian group G. The concept of grading is useful if tensor products or algebraic products are relevant (both are in the present application) and if the factors are to carry into the products some "memory" of their origin[3]. This memory is supplied by the elements of the group G.

To simplify the object $\mathcal{Q} = \{\mathcal{O}, \mathcal{L}, \sigma, \omega, \gamma, e\}$ one might consider taking the direct sum of the spaces $\mathcal{O}$ and $\mathcal{L}$ and work with a single linear space, but this would irretrievably mix the observables and the generators. To have a reversible mixing, these objects must carry some memory of their origin, which points to grading. To this end, we take a group with at least two elements, $G = \{I, \varpi, \ldots\}$, and define a new linear space $\mathcal{B}$ as the direct sum

$$\mathcal{B} \stackrel{\text{def}}{=} I\mathcal{O} \oplus \varpi\mathcal{L}. \tag{29}$$

(The other option, $\varpi\mathcal{O} \oplus \mathcal{I}\mathcal{L}$, is not possible, since the space $\mathcal{O}$ has an algebraic unit, e, which must go with the group unit I for compatibility). The reversal operator can now be transferred from the spaces $\mathcal{O}, \mathcal{L}$ to the group G by expressing the relations (27), (28) as

$$\mathbf{R}\{I, \varpi, \ldots\} = \{I, -\varpi, \ldots\}, \tag{30}$$

which enables us to retrieve the spaces $\mathcal{O}, \mathcal{L}$ from the space $\mathcal{B}$. Hence, the algebraic object $\mathcal{Q}$ may also be written as

$$\mathcal{Q} = \{\mathcal{B}, \mathbf{R}, \sigma, \omega, \gamma, e\}.$$

This form has the advantage of being based on a single linear underlying space, $\mathcal{B}$, but that space is graded. The connection between the two equivalent forms of $\mathcal{Q}$ is established by the involution $\mathbf{R}$, which is a structural element. While it may be obvious that the group G consists of only two element, this will soon be implied by composability.

[3]As an example of such "memory", consider the LTM dimensions of mechanics: to every mechanical object is associated a dimension $L^{\alpha}T^{\beta}M^{\gamma}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$, and $\mathbb{Z}$ is the additive group of integers, so that the Abelian group in question is $G = \mathbb{Z}^{\nvDash}$.

2.11. Inner derivations

In both classical and quantum mechanics, the observables play an additional role — they generate all the infinitesimal automorphisms of the algebraic product. This is equivalent to saying that all derivations are inner. In classical mechanics, this takes place by way of the Poisson bracket,

$$f \to f + \varepsilon \{g, f\} \equiv f + \varepsilon \left(\frac{\partial g}{\partial p} \frac{\partial f}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial p} \right),$$

in quantum mechanics, by way of the commutator,

$$F \to F + \varepsilon \frac{1}{\mathrm{i}} [G, F] \equiv F - \varepsilon \mathrm{i} (GF - FG).$$

This double role of the observables is referred to as the *equivalence of observables and generators.*

The abstract model for classical mechanics is $\mathcal{L} = \mathcal{O}$, where the products γ and ω coincide and are realized by the Poisson bracket. In quantum mechanics there are two choices. One could take $\mathcal{L} = \mathcal{O}$, and interpret γ and ω as the commutator divided by the imaginary unit, or, alternatively, one could take $\mathcal{L} = \mathrm{i}\mathcal{O}$ (the generators would then be anti-Hermitian matrices), and interpret γ and ω as the commutator. We shall initially take the latter viewpoint. Note: The identification of the spaces $\mathcal{O}$ and $\mathcal{L}$ adjoins the unit e to the Lie algebra, where it plays the role of an inconsequential central element, i.e., $e\omega\mathcal{L} = \{0\}$.

It is extremely suggestive that the equivalence of observables and generators strongly characterizes mechanics (both classical and quantum). Indeed, it seems impossible to find a non-trivial analogy anywhere else in mathematics — with one apparent exception: there seems to be a non-mechanical counter-example in three-dimensional Euclidean geometry. To construct it, let the space $\mathcal{O}$ be defined as the space of Euclidean vectors. The space $\mathcal{L}$ would then consist of the generators of the rotation group, an infinitesimal rotation being written in the form

$$\vec{r} \to \vec{r} + \varepsilon \vec{g} \times \vec{r}.$$

The space $\mathcal{O}$ (of polar vectors $\vec{r}$) and the space $\mathcal{L}$ (of axial vectors $\vec{g}$) are both 3-dimensional linear spaces. Hence, they are isomorphic if we fix the orientation, thus eliminating the distinction between polar and axial vectors. Yet, notwithstanding appearances, this is not a true counter-example due to the local isomorphism between rotations and spin, $SO\,(3) \backsim SU\,(2)$ — all unitary groups being structurally quantum mechanical[4].

[4] In Part III of this work, the exhaustive classification of the realizations of quantal algebra will yield the group $SO\,(6)$ as a second counter-example, but again, only apparently so, due to the local isomorphism $SO\,(6) \sim SU\,(4)$ under a fixed orientation.

3. *The Quantal Algebra*

The set of abstract concepts extracted from classical and quantum mechanics and related to postulate (B1) is now to be worked into an axiomatic algebraic structure (the quantal algebra). The first step has already been taken with the definition of the object

$$\mathcal{Q} = \{\mathcal{O}, \mathcal{L}, \sigma, \omega, \gamma, e\} = \{\mathcal{B}, \mathbf{R}, \sigma, \omega, \gamma, e\}\,,$$

which axiomatically incorporates the real linearity of the set of observables, the existence of a symmetric product, and the reversibility of all automorphisms of this product. Two constructive properties remain to be incorporated: Composability, and the equivalence of observables and generators (the reality conditions are restrictive, not constructive). We take the former as axiomatic, as it is strictly conceptual. Thus, we postulate the possibility of unrestricted **pairwise composition** of all instances of objects $\mathcal{Q}$ within each composition class $\{\mathfrak{T}, \bigcirc\}$, the composite objects being new instances of the same structure.

Referring to relations (23) through (26), the preservation of linearity under composition implies that the spaces $\mathcal{O}_{12}$ and $\mathcal{L}_{12}$ are to be defined by tensor products. This requirement is more naturally expressed in terms of single graded spaces, $\mathcal{B}_i$, than in terms of pairs of spaces, $\mathcal{O}_i$, $\mathcal{L}_j$, as

$$\{\mathcal{B}_{ij}, \mathbf{R}, \ldots\} = \{\mathcal{B}_i \otimes \mathcal{B}_j, \mathbf{R}, \ldots\}\,,$$

which implies

$$\mathcal{O}_{ij} = (\mathcal{O}_i \otimes \mathcal{O}_j) \oplus \varpi^2 (\mathcal{L}_i \otimes \mathcal{L}_j)\,, \tag{31}$$

$$\mathcal{L}_{ij} = (\mathcal{O}_i \otimes \mathcal{L}_j) \oplus (\mathcal{L}_i \otimes \mathcal{O}_j)\,. \tag{32}$$

Hence, $\varpi^2 = I$, so that $G = \{I, \varpi\}$ is a group of order two, as expected. But the expressions (31) and (32) are not bilinear if the spaces $\mathcal{O}_i$ and $\mathcal{L}_i$ are mutually unrelated. For a linear transformation in $\mathcal{O}_i$ to propagate to the composite space $\mathcal{O}_{ij}$ it must also propagate to $\mathcal{L}_i$, and vice-versa. Hence, linearity-preserving composition requires that linear transformations in $\mathcal{O}_i$ act simultaneously in $\mathcal{L}_i$, and vice-versa, which implies that the spaces $\mathcal{O}_i$ and $\mathcal{L}_i$ are rigidly related by a fixed linear isomorphism. Let's denote such an isomorphism by L_i. Thus,

$$\mathcal{L}_i = L_i \mathcal{O}_i\,. \tag{33}$$

As we need only finite dimensional spaces, this implies

$$\dim(\mathcal{O}_i) = \dim(\mathcal{L}_i)\,.$$

We recognize in this conclusion the equivalence of observables and generators — which thus appears as a theorem, and need not be postulated. We shall have to find all inequivalent solutions for the isomorphism L, but we first note that its existence alone simplifies and tightens the structure of the object $\mathcal{Q}$. Indeed, the elimination of the space $\mathcal{L}$ transfers the products ω and γ to the space $\mathcal{O}$.

We first transfer the Lie product $H = F\omega G$ from $\mathcal{L}$ to $\mathcal{O}$ by the substitutions $H = Lh$, $F = Lf$, $G = Lg$, which yield

$$Lh = (Lf)\,\omega\,(Lg)\ .$$

This relation implies that h is a bilinear function of f and g, i.e., it is a product in the space of observables. We denote it by α, its defining identity being

$$L\,(f\alpha g) \equiv (Lf)\,\omega\,(Lg)\,. \tag{34}$$

The space of observables is now equipped with two algebraic products, $\{\mathcal{O}, \sigma, \alpha, e\}$. Relations (21), (22) and (19) imply that the structure $\{\mathcal{O}, \alpha, e\}$ is also a Lie algebra, the unit e being a central element,

$$f\alpha g + g\alpha f = 0, \tag{35}$$

$$(f\alpha g)\,\alpha h + (h\alpha f)\,\alpha g + (g\alpha h)\,\alpha f = 0, \tag{36}$$

$$f\alpha e = 0. \tag{37}$$

We next transfer the product γ from $\mathcal{L}\times\mathcal{O}$ to $\mathcal{O}\times\mathcal{O}$. Denoting the new product by γ', its defining identity is

$$H\gamma f = (Lh)\,\gamma f \equiv h\gamma' f.$$

Since the product α is structural, it must be preserved by the already introduced group of isomorphisms of the product σ, which implies that the operator $H\gamma$ is a derivation, i.e., it satisfies a Leibnitz identity with respect to α. In terms of γ',

$$h\gamma'\,(f\alpha g) = (h\gamma' f)\,\alpha g + f\alpha\,(h\gamma' g)\,,$$

but this is the same relation as the Jacobi identity (36) if γ' is proportional to α. Proportionality factors being irrelevant, it follows that $\gamma' = \alpha$. Finally, the Leibnitz identity (18) becomes

$$h\alpha\,(f\sigma g) = (h\alpha f)\,\sigma g + f\sigma\,(h\alpha g)\,. \tag{38}$$

This completes the transfer of the algebraic products from the two spaces $\mathcal{O}$ and $\mathcal{L}$ to the single space $\mathcal{O}$. The object $\mathcal{Q}$ is now defined as a structure $\{\mathcal{O}, \sigma, \alpha, e, L\}$ whose two products satisfy the identities (35) to (38), and all of whose instances fall into one or more composition classes, $\{\mathfrak{T}, \bigcirc\}$ — the composition of the space of observables being defined by the tensor product

$$\mathcal{O}_{ij} = \mathcal{O}_i \otimes \mathcal{O}_j\ . \tag{39}$$

The next task is to find all solutions for the operator L, which is the only remaining unknown in the object $\mathcal{Q}$.

From the expressions (31), (32), (33), (39) and $\varpi^2 = I$, one obtains

$$\begin{aligned} \mathcal{O}_{ij} &= \mathcal{O}_{ij} \oplus (L_i \otimes L_j)\, \mathcal{O}_{ij} \\ L_{ij}\mathcal{O}_{ij} &= (\mathcal{O}_i \otimes L_j\mathcal{O}_j) \oplus (L_i\mathcal{O}_i \otimes \mathcal{O}_j) \\ &= (I_i \otimes L_j)\, \mathcal{O}_{ij} \oplus (L_i \otimes I_j)\, \mathcal{O}_{ij}\,. \end{aligned}$$

Since the indices i and j label arbitrary objects $\mathcal{Q}_i \in \mathfrak{T}$, it follows from these identities that the operators L_i are proportional to the unit matrix,

$$L_i = I_i \hat{J}, \tag{40}$$

where $\hat{J}$ is a universal operator — in the sense that it acts in the same way on all observables in all objects in the composition class $\mathfrak{T}$ (multiplication by a constant being such an operator). Its square is equivalent to multiplication by a real number,

$$\hat{J}^2 = -a, \tag{41}$$

where $a \in \mathbb{R}$, the minus sign being conventional.

We note that if the constant a does not vanish, it can be normalized to unit absolute value by simultaneously re-scaling $\hat{J}$ and the generators:

$$\begin{aligned} \hat{J} &\rightarrow \hat{J}/\sqrt{|a|}, \\ \mathcal{L}_i &\rightarrow \sqrt{|a|}\mathcal{L}_i. \end{aligned}$$

Hence, a has only three inequivalent values,

$$a \in \{+1, 0, -1\}\,. \tag{42}$$

In the computations which follow, we use the notations $f_i, g_i, ... \in \mathcal{O}_i$, $f_i \otimes f_j \in \mathcal{O}_{ij}$, etc. Then, by relations (31), (34), (40) and (41), we have

$$\begin{aligned} (f\sigma g)_{ij} &= f_{ij}\sigma_{ij}g_{ij} \\ &= (f_i \otimes f_j)\, \sigma_{ij}\, (g_i \otimes g_j) \\ &= (f\sigma g)_i \otimes (f\sigma g)_j + a\, (f\alpha g)_i \otimes (f\alpha g)_j\,. \end{aligned} \tag{43}$$

Similarly, for the product alpha in the composite algebra we get

$$\begin{aligned} (f\alpha g)_{ij} &= f_{ij}\alpha_{ij}g_{ij} \\ &= (f_i \otimes f_j)\, \alpha_{ij}\, (g_i \otimes g_j) \\ &= (f\sigma g)_i \otimes (f\alpha g)_j + (f\alpha g)_i \otimes (f\sigma g)_j\,. \end{aligned} \tag{44}$$

The symmetry conditions (1) and the antisymmetry condition (35) in the composite algebra $\mathcal{O}_{ij}$ are easily verified. Indeed, we note that the expression (43) is symmetric and the expression (44) antisymmetric with respect to the exchange $f \leftrightarrow g$ simultaneously performed in both $\mathcal{O}_i$ and $\mathcal{O}_j$. One also easily verifies the identities $f_{ij}\sigma_{ij}e_{ij} = f_{ij}$ and $f_{ij}\alpha_{ij}e_{ij} = 0$.

The relations (43) and (44) give the general expressions for the sigma and alpha products in the quantal algebra $\mathcal{Q}_{ij}$ as functions of these products in the component algebras $\mathcal{Q}_i$, $\mathcal{Q}_j$, but these expressions still have to satisfy the Jacobi and Leibnitz identities. We impose these conditions in turn.

The Leibnitz condition in the composite algebra reads

$$\begin{aligned} 0 \quad = \quad & -(h_i \otimes h_j)\,\alpha_{ij}\,[(f_i \otimes f_j)\,\sigma_{ij}\,(g_i \otimes g_j)] \\ & +[(h_i \otimes h_j)\,\alpha_{ij}\,(f_i \otimes f_j)]\,\sigma_{ij}\,(g_i \otimes g_j) \\ & +(f_i \otimes f_j)\,\sigma_{ij}\,[(h_i \otimes h_j)\,\alpha_{ij}\,(g_i \otimes g_j)]\,. \end{aligned}$$

The algebraic manipulations needed to transform this identity into a simpler one consist of three steps:

1) Substitute into this relation the expressions for σ_{ij} and α_{ij} given by relations (43) and (44). This yields

$$\begin{aligned} 0 = & -(h\sigma\,(f\sigma g))_i \otimes (h\alpha\,(f\sigma g))_j - (h\alpha\,(f\sigma g))_i \otimes (h\sigma\,(f\sigma g))_j \\ & + a\,(h\sigma\,(f\alpha g))_i \otimes (h\alpha\,(f\alpha g))_j + a\,(h\alpha\,(f\alpha g))_i \otimes (h\sigma\,(f\alpha g))_j \\ & + ((h\sigma f)\,\sigma g)_i \otimes ((h\alpha f)\,\sigma g)_j - a\,((h\sigma f)\,\alpha g)_i \otimes ((h\alpha f)\,\alpha g)_j \\ & + ((h\alpha f)\,\sigma g)_i \otimes ((h\sigma f)\,\sigma g)_j - a\,((h\alpha f)\,\alpha g)_i \otimes ((h\sigma f)\,\alpha g)_j \\ & + (f\sigma\,(h\sigma g))_i \otimes (f\sigma\,(h\alpha g))_j - a\,(f\alpha\,(h\sigma g))_i \otimes (f\alpha\,(h\alpha g))_j \\ & + (f\sigma\,(h\alpha g))_i \otimes (f\sigma\,(h\sigma g))_j - a\,(f\alpha\,(h\alpha g))_i \otimes (f\alpha\,(h\sigma g))_j\ . \end{aligned}$$

2) Using the Leibnitz identity, expand all terms of the form $x\alpha\,(y\sigma z)$. This leaves mixed product expressions with the products α only inside the inner parentheses.

3) Extract all factors of the form $(x\alpha y)\,\sigma z$. Collecting the trilinear sigma products into associators, as defined by relation (2), and using the Jacobi identity, one obtains

$$\begin{aligned}
0 \;=\;& ((h\alpha f)\,\sigma g)_i \otimes \{[h,f,g] - af\alpha\,(g\alpha h)\}_j \\
&+ ((h\alpha g)\,\sigma f)_i \otimes \{[h,g,f] - ag\alpha\,(f\alpha h)\}_j \\
&+ \{[h,f,g] - af\alpha\,(g\alpha h)\}_i \otimes ((h\alpha f)\,\sigma g)_j \\
&+ \{[h,g,f] - ag\alpha\,(f\alpha h)\}_i \otimes ((h\alpha g)\,\sigma f)_j \,.
\end{aligned}$$

This is still a two-space relation, as it mixes the spaces $\mathcal{O}_i$ and $\mathcal{O}_j$. To reduce it to an algebraic identity in a single space, we take its cyclic sum with respect to the variables f_i, g_i, h_i. This eliminates the third and fourth terms (due to the Jacobi identity and to the fact that the cyclic sum of the associator vanishes identically) and consolidates the first two terms into a single tensor product:

$$\begin{aligned}
0 \;=\;& \left[\sum_{cyclic} ((h\alpha g)\,\sigma f)_i\right] \otimes \\
& \{[h,f,g] - af\alpha\,(g\alpha h) - [h,g,f] + ag\alpha\,(f\alpha h)\}_j \;.
\end{aligned}$$

Obviously, one of the factors must vanish identically. It cannot be the first one, as the special case of $f = e$ would imply $h\alpha g = 0$ for all h and all g, trivializing the solution. Hence, it is the second term which vanishes for all j. We thus have

$$[h,f,g] - [h,g,f] = a\,[f\alpha\,(g\alpha h) - g\alpha\,(f\alpha h)]\,.$$

This is a single-space identity, as desired, but it can be further simplified. Using the Jacobi identity and expanding the associators using the definition (2), this relation simplifies to

$$[f,h,g] = ah\alpha\,(g\alpha f)\,. \qquad (45)$$

This identity characterizes mechanics, both classical ($a = 0$) and quantum ($a \neq 0$). We refer to it as the **association identity**.

The Jacobi identity still remains to be imposed or verified. Expanding one term in equation (36), we get

$$\begin{aligned}
& [(f_i \otimes f_j)\,\alpha_{ij}\,(g_i \otimes g_j)]\,\alpha_{ij}\,(h_i \otimes h_j) \\
=\;& \Big[(f\sigma g)_i \otimes (f\alpha g)_j + (f\alpha g)_i \otimes (f\sigma g)_j\Big]\,\alpha_{ij}\,(h_i \otimes h_j) \\
=\;& [(f\sigma g)\,\sigma h]_i \otimes [(f\alpha g)\,\alpha h]_j + [(f\sigma g)\,\alpha h]_i \otimes [(f\alpha g)\,\sigma h]_j \\
&+ [(f\alpha g)\,\sigma h]_i \otimes [(f\sigma g)\,\alpha h]_j + [(f\alpha g)\,\alpha h]_i \otimes [(f\sigma g)\,\sigma h]_j \,.
\end{aligned}$$

By straightforward algebraic manipulations using the three identities (Jacobi, Leibnitz and association), one verifies that the cyclic sum of the first term on the right hand side vanishes, and so does the fourth. The cyclic sum of the second and third terms taken together vanishes by the Jacobi and Leibnitz identities alone. Hence, the Jacobi identity is satisfied without implying new conditions. The structure $\mathcal{Q}$ common to both mechanics is now fully determined. We call it a **quantal algebra**. While $a = 0$ characterizes quantum mechanics, it remains to be determined whether $a = 1$ or $a = -1$ characterizes quantum mechanics, and hence its generalization. To this end, it suffices to compute the two sides of the association identity for observables represented by Hermitian operators, where $f\sigma g = \frac{1}{2}(fg + gf)$, and $f\alpha g = \frac{1}{2\mathrm{i}}(fg - gf)$. For the associator we get

$$\begin{aligned} 4[f,g,h] &= (fg+gf)\,h + h\,(fg+gf) - f\,(gh+hg) - (gh+hg)\,f \\ &= fgh + gfh + hfg + hgf - fgh - fhg - ghf - hgf \\ &= gfh + hfg - fhg - ghf. \end{aligned}$$

Similarly, for the right hand side of relation (45) we get

$$\begin{aligned} 4g\alpha\,(h\alpha f) &= -\left[g\,(hf - fh) - (hf - fh)\,g\right] \\ &= -ghf + gfh + hfg - fhg. \end{aligned}$$

Hence

$$[f,g,h] = g\alpha\,(h\alpha f)\,,$$

which implies that the parameter value $a = 1$ in (45) characterizes quantum mechanics. Thus, there are two composition classes, $\mathfrak{T}_0$ (for $a = 0$) and $\mathfrak{T}_1$ (for $a = 1$).

We can now formulate the definition of a quantal algebra as a real structure independent of the still unknown operator $\hat{J}$.

Definition 2 *A* ***quantal algebra*** : $\mathcal{Q} = \{\mathcal{O}, \sigma, \alpha, e, a\}$, *where* $a \in \{1, 0\}$, *is a real linear space* $\mathcal{O}$ *equipped with two algebraic products which satisfy the following conditions*:

(a) The substructure $\{\mathcal{O}, \sigma, \mathrm{e}\}$ *is a symmetric algebra with a unit, (1), (6).*

(b) The spectrum of every element of $\mathcal{O}$ *is real.*

(c) The substructure $\{\mathcal{O}, \alpha, \mathrm{e}\}$ *is a Lie algebra, (35), (36).*

(d) The Lie product, α, *is a derivation with respect to the product* σ, *i.e., the two products are related by the Leibnitz identity (38).*

(e) The two products are also related by the association identity (45), where a $(= 0$ *or* $1)$ *is the composition class parameter.*

To put these results in perspective we briefly review their derivation:

(1) The subalgebra $\{\mathcal{O}, \sigma, e\}$ is axiomatic, having been directly extracted from classical and quantum mechanics.

(2) The composability requirement is also axiomatic, but has the flavour of a general principle. It can be viewed as a *meta-postulate*, as it does not specify a

substructure for the quantal algebra, but states that, whatever that algebra may be, it must be such that pairwise composition be possible.

(3) All other substructures of the quantal algebra, i.e., the Lie algebra and the Leibnitz and association identities, are derived concepts and consequences of these two axioms.

Hence, keeping these points in mind, the definition (2) can be compacted to a simple statement: *A quantal algebra is a composable real commutative algebra.*

The comparative study of classical and quantum mechanics having led to the development of the quantal algebra, we no longer need to refer to either original theory. The primary object of study in the sequel will be the composition class $\mathcal{T}_1$, but we shall remain aware of $\mathcal{T}_0$ as a limiting case. We may thus impose on the quantal algebra additional restrictions characteristic of quantum mechanics (in addition to $a = 1$). Thus, noting that the unitary groups are semi-simple, while the canonical groups are not, we shall be primarily concerned with semi-simple quantal algebras.

Definition 3 *A* ***semi − simple quantal algebra*** $\{\mathcal{O}, \sigma, \alpha, e, a\}$ *is a quantal algebra whose Lie subalgebra* $\{\mathcal{O}, \alpha\}$ *is semi-simple.*

The value of this restriction is obvious: it puts at our disposal Cartan's structure theory of semi-simple Lie groups — even though this theory remains to be extended to include the product sigma.

This completes the construction of the real abstract structure of mechanics formulated in the definition (3). The operator $\hat{J}$ plays no explicit role in this definition because it is a complexification unit. This is obvious from relation (41) for quantum mechanics, i.e., for $a = 1$, since the solution for $\hat{J}$ is then multiplication by the imaginary unit i. We shall come back to $\hat{J}$ in a later part of the present work, where we study the complexification of the quantal algebra.

References

[1] S. Weinberg, *Dreams of a Final Theory*, Pantheon Books, New York (1992).

[2] E. Grgin and A. Petersen, Commun. Math. Phys. **50** (1976) 177.

[3] E. Grgin and G. Sandri, Fizika B (Zagreb) **5** (1996) 141, and Fizika B (Zagreb) **6** (1997) 1.

[4] G. M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, Kluwer Academic Publishers (1994).

[5] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press (1995).

[6] F. Gürsey and Chia-Hsiung Tze, *On the Role of Division, Jordan and Related Algebras in Particle Physics*, World Scientific, Singapore (1996).

[7] S. Okubo, *Introduction to Octonion and other Non-Associative Algebras in Physics*, Cambridge University Press, (1995).

SUŠTINSKI RELATIVISTIČKA KVANTNA TEORIJA
I Dio. ALGEBRA OPSERVABLI

Ovo je prvi članak programa kojim se želi poopćiti kvantna mehanika zadržavajući njenu strukturu u biti nedirnutu, ali razvijajući Hilbertov prostor na novom sustavu brojeva koji je mnogo bogatiji od polja kompleksnih brojeva. Taj se sustav naziva "Kvantionskom algebrom". On je osmerodimenzijski kao i algebra oktoniona, ali je, za razliku, asocijativan. To nije diobena algebra već je "gotovo" takva (to u nekom smislu postaje jasno kad proučimo taj dio). Ta algebra ima minimalna svojstva potrebna za gradnju Hilbertovog prostora koji omogućuje kvantnomehanička tumačenja (poput prijelaznih vjerojatnosti), a, povrh toga, sadrži strukturu prostora-vremena Minkowskog. Stoga je kvantna teorija zasnovana na kvantionima suštinski relativistička. Algebra kvantiona je otkrivena u dva koraka. Prvi je pažljiva analiza apstraktne strukture kvantne mehanike (prvi dio ovog rada), a drugi je klasifikacija svih stvarnih realizacija te apstraktne strukture (ostatak ovog rada i niz dodatnih članaka). Klasifikacija pokazuje da postoje samo dvije realizacije. Jedna je standardna kvantna mehanika, a druga suštinski relativistička generalizacija. U ovom se radu razvija apstraktna algebra opservabli.