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THE ENERGY OF A TOPOLOGICAL BLACK HOLE

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We calculate the energy distribution of a topological black hole by using the Einstein's and Tolman's energy-momentum pseudotensors. All the calculations are performed with approximation in quasi-Cartesian coordinates. We get the same result for both the Einstein's and Tolman's prescriptions. The energy distribution of the topological black hole depends on the mass M, electric charge Q and cosmological constant Λ.

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1. Introduction

The general theory of relativity is an excellent theory of space, time and gravitation and has been supported by experimental evidences with flying colours, but some of its features are not without difficulties. For instance, the subject of energy-momentum localization has been a problematic issue since the outset of this theory.

A large number of definitions of the gravitational energy have been given since. Some of them are coordinate independent and other are coordinate-dependent. It is possible to evaluate the energy distribution and momentum by using various energy-momentum pseudotensors. There lies a dispute on the importance of non-tensorial energy-momentum complexes whose physical interpretations have been questioned by a number of physicists, including Weyl, Pauli and Eddington. Also, there exists an opinion that the energy-momentum pseudotensors are not useful to get meaningful energy distribution in a given geometry.

Several examples of particular space-times (the Kerr-Newman, the Einstein-Rosen and the Bonnor-Vaidya) have been investigated and different energy-momentum pseudotensors are known to give the same energy distribution for a given space-time [1–6]. Aguirregabiria, Chamorro and Virbhadra [7] showed that several energy-momentum pseudotensors coincide for any Kerr-Schild–class metric. Xulu obtained interesting results about the energy distribution of a charged dilaton black hole [8] and about the energy associated with a Schwarzschild black hole in a magnetic universe [9]. Also, recently, Xulu [10] obtained the total energy of a model of universe based on the Bianchi type I metric. The author calculated the energy distribution of a dilaton dyonic black hole in the Tolman's prescription [11] and obtained an acceptable result. Also, this result is the same as the result obtained by I-Ching Yang, Ching-Tzung Yeh, Rue-Ron Hsu and Chin-Rong Lee [12] in the Einstein's prescription. Recently, Virbhadra [13] showed that different energy-momentum pseudotensors give the same and reasonable results for many space-times.

A topological black hole can be defined as a space-time whose local properties are trivial (constant curvature) but its causal global structure is that of a black hole. The topological structure of the event horizon of a black hole is an interesting subject in black-hole physics. It is generally believed that a black hole in the four-dimensional space-time is always with a spherical topology. The event horizon of a black hole has the topology S^2 and this problem was clarified by Friedman, Schleich and Witt [14].

In the recent years, there has been a growing interest in these black holes with nontrivial topological structures (topological black holes) in the asymptotically anti-de Sitter space. The investigations are mainly based on the Einstein-Maxwell theory with a negative cosmological constant.

In this paper we compute the energy distribution of a topological black hole by using both the Einstein's and Tolman's prescriptions. We obtain the same result in both prescriptions. We also make a discussion of the results. We use the geometrized units ($G = 1, c = 1$) and follow the convention that the Latin indices run from 0 to 3.

2. The energy distribution

The static solutions of Einstein-Maxwell equations with a cosmological constant are

$$\mathrm{d}s^2 = \left(k - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 - \left(k - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega_k^2, \tag{1}$$

where M, Q and Λ are the mass, the electric charge of the black hole and the cosmological constant (which is negative), respectively. Also, $\mathrm{d}\Omega_k^2$ is the line element of a two-dimensional hypersurface Σ with a constant curvature

$$\mathrm{d}\Omega_k^2 = \begin{cases} \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2 & \text{for } k = 1, \\ \mathrm{d}\theta^2 + \theta^2\mathrm{d}\varphi^2 & \text{for } k = 0, \\ \mathrm{d}\theta^2 + \sinh^2\theta\mathrm{d}\varphi^2 & \text{for } k = -1. \end{cases} \tag{2}$$

For $k = 1$, the metric (1) describes the Reissner-Nordström-anti-de Sitter black holes. The event horizon of the black hole has the 2-sphere topology S^2, and the topology of the space-time is $R^2 \times S^2$. For $k = 0$, if we identify the coordinates

θ and φ with certain periods, the topology of the event horizon is that of a torus and the space-time has the topology $R^2 \times T^2$. For $k = -1$, the surface Σ is a 2-dimensional hypersurface with constant negative curvature. The topology of the space-time is $R^2 \times H_g^2$ [15], where H_g^2 is the topology of the surface Σ.

We obtain the energy distribution of a topological black hole given by (1) in the case $k = 1$ both in the Einstein's and Tolman's prescriptions.

The Einstein's energy-momentum pseudotensor [16] is given by

$$\Theta_i^k = \frac{1}{16\pi} H_i^{kl},_l \,, \tag{3}$$

where

$$H_i^{kl} = -H_i^{lk} = \frac{g_{in}}{\sqrt{-g}} \left[-g \left(g^{kn} g^{lm} - g^{\ln} g^{km} \right) \right]_m \,. \tag{4}$$

Θ_0^0 and Θ_α^0 are the energy and the momentum components, respectively.

The energy-momentum pseudotensor Θ_i^k satisfies the local conservation laws

$$\frac{\partial \Theta_i^k}{\partial x^k} = 0 \,. \tag{5}$$

The Tolman's energy-momentum pseudotensor [17] is given by

$$\Upsilon_i^k = \frac{1}{8\pi} U_i^{kl},_l \,, \tag{6}$$

where Υ_0^0 and Υ_α^0 are the energy and momentum components.

We have

$$U_i^{kl} = \sqrt{-g}(-g^{pk} V_{ip}^l + \frac{1}{2} g_i^k g^{pm} V_{pm}^l), \tag{7}$$

with

$$V_{jk}^i = -\Gamma_{jk}^i + \frac{1}{2} g_j^i \Gamma_{mk}^m + \frac{1}{2} g_k^i \Gamma_{mj}^m. \tag{8}$$

The energy-momentum pseudotensor Υ_i^k also satisfies the local conservation laws

$$\frac{\partial \Upsilon_i^k}{\partial x^k} = 0. \tag{9}$$

Both the Einstein's and Tolman's energy-momentum pseudotensors give the correct result if the calculations are carried out in quasi-Cartesian coordinates. We transform the line element (1) to quasi-Cartesian coordinates t, x, y, z according to

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta \,, \tag{10}$$

and

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}. \tag{11}$$

The line element (1) becomes

$$\mathrm{d}s^2 = \left(k - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{12}$$
$$-\frac{1}{r^2}\left(\left(k - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda r^2\right)^{-1} - 1\right)(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)^2.$$

The only required components of H_i^{kl} in the calculation of the energy are the following

$$H_0^{01} = \frac{2}{3}\frac{x\Pi}{r^4}, \qquad H_0^{02} = \frac{2}{3}\frac{y\Pi}{r^4}, \qquad H_0^{03} = \frac{2}{3}\frac{z\Pi}{r^4}. \tag{13}$$

Also, the only required components of U_i^{kl} in the calculation of the energy are the following

$$U_0^{01} = \frac{1}{3}\frac{x\Pi}{r^4}, \qquad U_0^{02} = \frac{1}{3}\frac{y\Pi}{r^4}, \qquad U_0^{03} = \frac{1}{3}\frac{z\Pi}{r^4}. \tag{14}$$

In the relations (13) and (14), we denote by Π

$$\Pi = 6Mr - 3Q^2 + \Lambda r^4. \tag{15}$$

The components of the pseudotensors H_i^{kl} and U_i^{kl} were calculated with the program Maple GR Tensor II Release 1.50.

We obtain that the energy and momentum in the Einstein's prescription are given by

$$P_i = \iiint \Theta_i^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3. \tag{16}$$

Also, in the Tolman's prescription, the energy and momentum are given by

$$P_i = \iiint \Upsilon_i^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3. \tag{17}$$

Using the Gauss's theorem, we get in the Einstein's prescription

$$P_i = \frac{1}{16\pi}\iint H_i^{0\alpha} n_\alpha \mathrm{d}S, \tag{18}$$

or, in the Tolman's prescription,

$$P_i = \frac{1}{8\pi}\iint U_i^{0\alpha} n_\alpha \mathrm{d}S. \tag{19}$$

In (17) and (18) $n_\alpha = (x/r, y/r, z/r)$ are the components of a normal vector over an infinitesimal surface element $\mathrm{d}S = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$.

We obtain the energy distribution of the topological black hole in the Einstein's prescription by using (13), (15), (16) and by applying the Gauss's theorem (18)

$$E(r) = M - \frac{Q^2}{2r} + \frac{1}{6}\Lambda r^3. \tag{20}$$

Also, in the Tolman's prescription, we calculate the energy distribution by using (14), (15), (17) and applying the Gauss's theorem (19). The result is the same as the result obtained in the Einstein's prescription

$$E(r) = M - \frac{Q^2}{2r} + \frac{1}{6}\Lambda r^3. \tag{21}$$

The energy distribution depends on the mass M, electric charge Q and cosmological constant Λ and is shared both by the interior and the exterior of the black hole. We obtain the same result in both prescriptions.

3. *Discussion*

The subject of energy-momentum localization has been associated with much debate. Bondi [18] gave his opinion that a non-localizable form of energy is not admissible in relativity so its location can in principle be found. The energy-momentum pseudotensors are not tensorial objects and one is compelled to use "Cartesian coordinates". For this reason, this subject was not considered seriously for a long time and was re-opened by the results obtained by Virbhadra, Rosen, Chamorro and Aguirregabiria. Some interesting results which have been found recently [7–8,10–13] lend support to the idea that the several energy-momentum pseudotensors can give the same and acceptable result for a given space-time. Also, in his recent paper, Virbhadra [13] emphasized that though the energy-momentum complexes are non-tensors, they do not violate the principle of general covariance as the equations describing the local conservation laws with these objects are covariant. Chang, Nester and Chen [19] showed that there exists a direct relationship between pseudotensors and quasilocal expressions, because every energy-momentum pseudotensor is associated with a legitimate Hamiltonian boundary term.

We obtain the same result for the energy distribution in both Einstein's and Tolman's prescriptions. This result lends support to the idea that the Einstein's and Tolman's energy-momentum pseudotensors can give the same result for a given space-time. Also, this is in some way an expected result because the Einstein's and Tolman's energy-momentum pseudotensors are equivalent in import, although different in form. The energy distribution depends on the mass M, electric charge Q and cosmological constant Λ. Both prescriptions provide the same energy in the exterior of the black hole which is given by $E(r) = -(Q^2/2r) + \frac{1}{6}\Lambda r^3$.

When $\Lambda = 0$, the line element (1) describes the spherically-symmetric solutions, and from (20) and (21) the energy distribution becomes $E(r) = M - Q^2/2r$. This is the expression of the energy in the case of the Reissner-Nordström field. The result is the same as that obtained by Chamorro and Virbhadra [20] for a charged dilaton black hole based on the Garfinkle-Horowitz-Strominger [21] (GHS) solutions when the coupling parameter $\beta = 0$. We remark that because the cosmological constant Λ is negative, the term $\frac{1}{6}\Lambda r^3$ (which is negative) in expressions for the energy distribution given by (20) and (21), respectively, is predominant. If $Q = 0$ and $\Lambda = 0$, the entire energy is confined to its interior with no energy shared by the exterior of the black hole. The energy distribution is $E(r) = M$. This result is similar to the case of the Schwarzschild black hole.

The energy distribution given by $E(r) = M - (Q^2/2r) + \frac{1}{6}\Lambda r^3$ can be interpreted as the “effective gravitational mass” that a neutral test particle “feels” in the Reissner-Nordström-anti-de Sitter space-time.

Also, the concept of a black hole lends support to the idea that the gravitational energy is localizable to finite regions.

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