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EIGENVALUES OF MULTITERM INVERSE-FRACTIONAL-POWER POTENTIALS

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It is shown that eigenspectra corresponding to various interactions with three- and two-term inverse fractional power are, under certain conditions, analytically tractable either by using the solution of the biconfluent Heun's equation or by using the quasi-exactly solvable sl(2) symmetry approach.

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1. *Introduction*

When the whole set of eigenfunctions and eigenvalues of the Schrödinger equation are accessible by purely algebraic means, the problem is said to be exactly solvable (ES) which usually is admitted to underline certain type of group-symmetry structure.

The problem becomes quasi-exactly sovable (QES) when only part of it can be reached algebraically, while the remaining part must be handled by numerical means implying also existence of some dynamical algebraic structure.

There is an intermediate situation, when exact solvability can be obtained only after specific constraints have been imposed on the parameters of the potentials, and the problem in this case is said to be conditionally exactly solvable (CES).

The first part of this paper serves to introduce the use of the "mixing function" formalism in order to examine the structure of the eigenspectrum of a three-term potential $V(A, B, C; x)$ involving inverse fractional powers with three strength parameters A, B and C.

The discussion is then extended to the case of quasi-exact solvability which allows to recover and confirm some earlier results obtained by other authors.

When one of the parameters is removed from the theory, then three types of two-term potentials with inverse fractional power will result, i.e., $V(A,C;\,x), V(B,C;\,x)$ and $V(A,B;\,x)$. Two of them can be approached in the frame of the present discussion.

The physical relevance of the potentials with the inverse fractional powers involving two and three terms have already been mentioned by some authors, for instance, the possibility of construction of double-well potentials [1], the existence of hidden symmetry sl(2) [2], the Coulomb correlation problems in atomic and molecular physics [3], etc.

2. *Formulation*

Consider the system of two coupled first-order differential equations which in matrix form is

$$\phi' + F\phi = 0\,, \qquad \phi = (\phi_1, \phi_2)^+\,, \qquad F = \begin{pmatrix} -u_1 & d_1 \\ 0 & u_2 \end{pmatrix}. \tag{1}$$

The details of this formulation can be found in Ref. [4]. In principle, the functions u_1, u_2 and d_1 may be any analytic functions.

The "mixing function" is defined as

$$\phi_1 = X\phi_2\,. \tag{2}$$

The compatibility condition must be

$$d_1 = -X' + (u_1 + u_2)X\,. \tag{3}$$

Differentiating (1) and making use of (3) leads to the following coupled equation

$$\phi_1'' - (u_1^2 + u_1')\phi_1 - [X'' - 2u_2X' + [(u_2^2 - u_2') - (u_1^2 + u_1')]X]\phi_2 = 0 \tag{4}$$

If E is the eigenvalue, then it can be seen that from (4), the Schrödinger equation will come out provided X be the solution of the second-order differential equation

$$X'' - 2u_2X' + [(u_2^2 - u_2') - (u_1^2 + u_1') - E]X = 0 \tag{5}$$

and

$$\phi_1'' - (u_1^2 + u_1')\phi_1 = E\phi_1 \tag{6}$$

By construction and up to a constant of normalisation, the set of eigenfunctions can be written as

$$\phi_{1,n} = X_n \exp(-\int u_2\,\mathrm{d}x)\,. \tag{7}$$

In other words, the problem of investigating the exact solvability of the Schrödinger equation (6) is now reduced to the search for the exact solutions of Eq. (5).

Furthermore, as the potential $V^+ = u_1^2 + u_1'$ defined in (6) is supersymmetric, it has a "partner" potential defined by $V^- = -u_1^2 - u_1'$ which, in order to be more complete, will now be included in the theory.

Consider again the system (1) in which

$$\bar{\phi}' + \bar{F}\bar{\phi} = 0\,, \qquad \bar{\phi} = (\bar{\phi}_1, \phi_2)^+\,, \qquad \bar{F} = \begin{pmatrix} u_1 & \bar{d}_1 \\ 0 & u_2 \end{pmatrix} \tag{1b}$$

With the same reasoning as above, a second mixing function can also be defined

$$\bar{\phi}_1 = Y\phi_2\,. \tag{2b}$$

It can be shown that the link between these two functions is

$$Y' - (u_2 - u_1)Y = X \tag{8}$$

(see the proof in the Appendix).

This result means that if one of these functions is known, then the other one can always be inferred by solving this first order differential equation.

3. *The potential*

The following potential is under consideration

$$V(A,B,C;x) = \frac{A}{\sqrt{x}} + \frac{B}{x} + \frac{C}{\sqrt{x^3}}\,. \tag{9}$$

A, B and C are free parameters.

Although it is usually admitted that, in principle, this type of potentials is not exactly solvable, it will be shown in this paper that under certain conditions, the present approach may lead to a different point of view. The theory relies on the appropriate choice of the two functions u_1 and u_2. Let

$$u_2 = \frac{1}{4}\frac{b}{\sqrt{x}}\,, \qquad u_1 = u_2 + k \tag{10}$$

b and k are paremeters.

The couple of potentials defined above now reads

$$V(A,B,C;x) = \frac{1}{2}\frac{kb}{\sqrt{x}} + \frac{1}{6}\frac{b^2}{x} \mp \frac{1}{8}\frac{b}{\sqrt{x^3}}$$

that is

$$A = \frac{1}{2}\,kb, \qquad B = \frac{1}{6}\,b^2, \qquad C = \mp\frac{1}{8}\,b\,,$$

so that the number of independent parameters in now reduced to two. In this case, the equation (5) become tractable with the following double transformation

$$x = t^2, \qquad X = t^2 \exp\left(\frac{1}{4}at^2 + bt\right)\, h(t) \tag{11}$$

a and b are arbitrary quantities, but it will be seen below that they can be expressed in terms of the original parameters b and k and $h(t)$ is the unknown function which must be determined.

Substituing (11) in (5), and after some algebra, one can verify that this function must be solution of the following second-order differential equation (the notations are $h' = \mathrm{d}h/\mathrm{d}t$, etc.)

$$th'' + (at^2 + bt + 3)h' + [(\frac{1}{4}a^2 - 4k^2 - 4E)t^3 + (\frac{1}{2}ab - 2kb)t^2 + 2at + 2b]h = 0 \tag{12}$$

4. *The Heun's biconfluent equation*

This type of equation has been recently investigated by Exton [1] who considered the second-order differential equation

$$ty'' + (a_1t^2 + b_1t + 3)y' + (a_0t^2 + 2a_1t + 2b_1)y = 0 \tag{13}$$

where a_1, b_1 and a_0 are free parameters. The interesting point is that it has been shown that this equation is exactly solvable provided these parameters are subject to a specific constraint

$$\left(\frac{a_0}{a_1}\right)^2 - \left(\frac{a_0}{a_1}\right) b = -(n+1)\,a_1, \qquad n = 0, 1, 2, \ldots \tag{14}$$

The solution is then

$$y = \frac{\mathrm{d}Y_n}{\mathrm{d}t}$$

where Y_n is represented by a linear expansion in terms of the powers of the variable t.

The detais of the analytical form of this expansion, which will not be displayed here, can be found in Ref. [1] (see for instance relation (13) there).

Equation (14) leads to a rather interesting situation where the structure of the eigenspectra become accessible from it.

In fact, comparing (12) with (13), it can be seen that they are identical if

$$\begin{array}{llll} \text{(a)} & E-k^2=\frac{1}{16}\,a^2\,, & \text{(b)} & a_0=\frac{1}{2}ab-2kb\,, \\ \text{(c)} & a_1=a\,, & \text{(d)} & b_1=b\,. \end{array} \tag{15}$$

Equation (15b) with the constraint (14) yields the following third-order algebraic equation for the unknown a

$$a^3-\frac{1}{4}\frac{b^2}{n+1}\,a^2+\frac{4k^2b^2}{n+1}=0\,, \tag{16}$$

from which the eigenspectrum can be inferred since E_n is given by (15a).

As this solution depends on two independent parameters, k and b, they can alwaus be chosen so that Eq. (16) leads to real, unique or multiple solutions. As the domain of the variable x is defined in $D[0,\infty]$, only negative solutions ($a<0$) can be retained since it leads to possible normalisation of the corresponding eigenfunction from (11).

Specifically, and as an example, we consider the first case (real, unique solution). Using the standard technique pertaining to this type of equation, and setting $k=mb$, $m>0$, m being any parameter, we find that the validity condition relative to this case is reduced to

$$1>\frac{b^2}{m^2}\,\frac{1}{(n+1)^2}\,10^{-3} \tag{17}$$

which depends on the ration b/m and n.

Interpretation: Consider two cases

$$1)\quad 1\le\frac{b}{m}<\sqrt{1000} \qquad\qquad 2)\quad \frac{b}{m}\ge 100$$

As n is positive, it is seen that the first case is always valid, meaning that the whole eigenspectrum can be reached analytically provided $a<0$. For instance, it can be verified that with $b=1$, $m=1$, the real solution a will always be negative. The second case means, the first three eigenvalue $n=0$, 1 and 2 are not accessible within the present approach, that is to say, this eigenspectrum will be split into two regions, the "upper" one, which can be reached (if $a<0$), and the "lower" one which is inaccessible.

A more complete description will be given later on with a specific example.

5. *The quasi-exact solvability*

When the set of parameters (A,B,C) are not subject to the contraints (10), supersymmetry is removed so that a more conventional approach must be considered. Let

$$\phi_1''-V(A,B,C;x)\phi_1=E\phi_1, \qquad \phi_1=Xg(x)\,. \tag{18}$$

where $g(x)$ is to be determined. We use now a transformation similar to (11), that is

$$x = t^2, \qquad X = t^2 \exp\left(\frac{1}{4}at^2 + \frac{1}{2}bt\right). \tag{19}$$

After substitution in (18), it can be verified that this function must be solution of the equation

$$\begin{aligned} & tg'' + (at^2 + bt + 3)g' \\ & + \left[\left(\tfrac{1}{4} - 4E\right)t^3 + \left(\tfrac{1}{2}ab - 4A\right)t^2 + \left(\tfrac{1}{4}b^2 + 2(a - 2B)\right)t + \tfrac{3}{2}b - 4C\right] g = 0 \end{aligned} \tag{20}$$

which becomes significant with the conditions

$$E = \frac{1}{16}a^2, \qquad \frac{1}{2}ab - 4A = 0, \qquad \frac{1}{4}b^2 + 2(a - 2B) = -an, \qquad n = 1, 2, \ldots. \tag{21}$$

since Eq. (20) can be expressed in terms of the generators J^+, J^- and J^0, of s1(2) symmetry, i. e.

$$\left[J^0 J^- + \left(3 + \frac{n}{2}\right) J^- + aJ^+ + bJ^0 + (3 - n)\frac{b}{2}\right] g = 4Cg. \tag{22}$$

Recalling that

$$J^+ = t^2 \frac{\mathrm{d}}{\mathrm{d}t} - nt, \qquad J^- = \frac{\mathrm{d}}{\mathrm{d}t}, \qquad J^0 = t\frac{\mathrm{d}}{\mathrm{d}t} - \frac{n}{2},$$

the preceding equation is quasi-exactly solvable since the differential operator in the left hand of (22) always preserves invariance of the finite dimensional subspace of the polynomials g [3]. Note that in the present case C must be seen as an "adjustabe parameter", i. e., for each n, it must be chosen appropriately. This is due to the type of three-term recurrence relation which would result from the usual expansion approach.

With these precautions, one may now extract some instructive results from the theory.

l) With the constraint (21, third relation), one can obtain the algebraic equation determining a, from which the eigenspectrum can be inferred

$$a^3 - \frac{1}{4}\frac{B}{n+2}a^2 + \frac{16A^2}{n+2} = 0. \tag{23}$$

Apart some slight modification in the notations in (23), it is in exact agreement with the result obtained previously in Ref. [2]. In the present discussion $l = 0$ (l is orbital quantum number). Extension to the case $l \neq 0$ is straightforward.

2) Consider now the two term potential $V(A,C;x)$ mentioned above ($B=0$) so that Eq. (21, third equation) reduces to

$$a = -\frac{1}{4}\,\frac{b^2}{n+2}\,,$$

(note that here $a<0$) from which

$$E_n = \frac{1}{4^{2/3}}\,\frac{A^{4/3}}{(n+2)^{2/3}}\,. \tag{24}$$

which is exactly the result obtained by Bose [3] who used the method of expansion with $l \neq 0$.

3) It would be instructive to examine simultaneously and on the same footing the cases of three- and two-term potentials, i.e., $V(A,B,C;x)$ and $V(A,C;x)$. The simplest way to achieve this is to return to the special case $k=1$ and $b=1$. For the first case, we obtain

$$E_n - 1 = \frac{0.062}{(n+1)^2}\,10^{-4} - \frac{0.159}{(n+1)^{4/3}} + \frac{0.101}{(n+1)^{2/3}}\,,$$

while for the second one

$$E_n = \frac{0.250}{(n+2)^{2/3}}\,.$$

This means that when $E_n \gg 1$ (the "upper" region), there may be some expected similarity for these two cases concerning the rate of decrease of E_n with increasing n.

4) Turning now to the second two-term potential $V(B,C;x)$ ($A=0$), the same reasoning remains valid so that Eq. (16) reduces to (since $k=0$)

$$a^2\left(a - \frac{1}{4}\,\frac{b^2}{n+1}\right) = 0\,. \tag{25}$$

Alternatively, if in (1) we set $u_1 = u_2$ and repeat the same procedure with the double transformation (11) where $h(t)$ is replaced by the function $f(t)$, then

$$tf'' + (at^2 + bt + 3)f' + \left[\left(\frac{1}{4}a^2 - 4E\right)t^3 + \left(\frac{1}{2}ab\right)t^2 + 2at + 2b\right] f = 0 \tag{26}$$

which in fact is the Heun's equation if

$$E = \frac{1}{16}\,a^2\,, \qquad a_0 = \frac{1}{2}\,ab\,, \qquad a_1 = a\,, \qquad b_1 = b\,,$$

and, therefore, yield the same solution as in (25). However, the difficulty here lies in the fact that for real values of b, the quantity a will always be positive, invalidating then the theory.

So, the only significant situation consists in the choice $b = \mathrm{i}c$ $(c > 0)$, i.e., an imaginary parameter. The eigenspectrum becomes

$$E_n = \frac{1}{16^2} \frac{C^4}{(n+1)^2} . \tag{27}$$

However, this choice does give rise to a non-hermitian Hamiltonian problem which is not our concern for the moment, although it can be noted recently that this domain of speculations has been considered by other authors (see for instance Ref. [5]).

6. *Conclusion*

The main results obtained in this work are summarised below

- For the quasi-exactly solvable problem, the method enables one to recover and confirm previous result of other authors concerning the three- and two-term potentials $V(A,B,C;x)$ and $V(A,C;x)$, A, B and C being arbitrary parameters.

- Under certain conditions, namely when the number of independent parameters of the three-term potential is reduced to only two, that is to say

$$A = \frac{1}{2}\, kb, \qquad B = \frac{1}{16}\, b^2, \qquad C = \mp \frac{1}{8}\, b\, ,$$

k, b being free parameters, then the three-term problem is shown to be supersymmetric and tractable with the "mixing function" formalism and using the solution of the biconfluent Heun's equation.

- The two-term potential of type $V(B,C;x)$, which cannot be handled with the usual QES approach (sl(2)), becomes tractable with the present formalism if $B = b^2/16$, $C = \pm b/8$ and and b is imaginary.

References

[1] H. Exton, J. Phys. A: Math. Gen. **28** (1995) 6739.

[2] Y. Brihaye, N. Devaux and P. Kopinski, Int. J. Mod. Phys. **10** 32 (1995) 4633.

[3] S. K. Bose, Nuovo Cimento B **109** (1994) 311.

[4] x. C. Cao, Int. J. Theor. Phys. **36** (1997) (no. 6) 1465.

[5] C. M. Bender and S. Boettcher, J. Phys. A: Math Gen. **31** (1998) L273 and references therein.

Appendix

1) In order to prove the relation (8), first note that, following the same procedure, it can be checked that the mixing function Y must be the solution of a differential equation similar to (4) but where the function u_1' is replaced by $-u_1'$. Using a more compact notation with the differential operator P defined by

$$P = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2u_2 \frac{\mathrm{d}}{\mathrm{d}x} + [(u_2^2 - u_2') - (u_1^2 - u_1')] \,, \quad \text{one obtains} \quad (P - E)Y = 0 \,. \tag{28}$$

If the result (8) is true, then by substituing into equation (4), one will be led to a third-order differential equation in terms of Y. After simplifications, it can be written in compact form

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} - (u_2 - u_1)\right] (P - E)Y = 0 \,,$$

which, from (28) is always true.

An alternative approach to this proof is to consider the technique of the "ladder operator" in supersymmetry wich also leads to the same result. For simplicity let us define the "ladder operator" as

$$A^{\pm} = -\frac{\mathrm{d}}{\mathrm{d}x} \pm u_1$$

and note that

$$A^+A^- = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (u_1^2 + u_1') \qquad \text{and} \qquad A^-A^+ = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (u_1^2 - u_1')$$

with the couple of eigenfunctions ϕ^+ and ϕ^- corresponding to the eigenvalue E_n.

ϕ^+ and ϕ^- must obey to the first-order coupled equations

$$A^-\phi_n^- = \sqrt{E_n}\,\phi_{n+1}^+ \qquad \text{and} \qquad A^+\phi_{n+1}^+ = \sqrt{E_n}\,\phi_n^- \,.$$

ϕ^+ and ϕ^- can be identified with the couple ϕ_1 and $\bar{\phi}_1$ considered in this work. If

$$\phi_1 = Y\phi_2, \qquad \bar{\phi}_1 = X\phi_2, \qquad \text{and} \qquad \phi_2 = \exp(\int u_2\,\mathrm{d}x)$$

then

$$\bar{\phi}_1 = const\, A^+\phi_1 \approx [Y' - (u_2 - u_1)]\, \exp(\int u_2\,\mathrm{d}x) \,,$$

from which follows the relation (8) in the text.

The interesting point here is to note that the above proof can be obtained independently of the supersymmetry, which implicitly means that this approach can be extended to a wider range of problems which may include non-supersymmetric systems.

Finally, for the special case $u_1 = u_2$, one obtains $X = Y'$, thus recovering the result obtained in the earlier work [4].

Although Eqs. (23) and (16) are similar (analytically speaking), they must be interpreted differently since in Eq. (23), the third parameter C must be chosen according to each n, and only the set of eigenfunctions in the "lower region" ($< n$) are accessible analytically.

In the second case, this role of adjustable parameter is played by k.

PROBLEM SVOJSTVENIH VRIJEDNOSTI ZA VIŠEČLANI POTENCIJAL S INVERZNIM RACIONALNIM POTENCIJAMA

Pokazuje se kako se, pod izvjesnim uvjetima, mogu analitički odrediti svojstvene vrijednosti za razna međudjelovanja s tri ili dva člana s inverznim racionalnim potencijama, primjenom bilo rješenja bikonfluentne Heunove jednadžbe, ili poluegzaktno rješivim pristupom sa sl(2) simetrijom.