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TEMPERATURE TRANSFORMATIONS IN RELATIVISTIC THERMODYNAMICS

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We analyse the problem of temperature transformation from a historical perspective and attempt to give an answer to the question: "Is the temperature of an object in relativistic motion lower, higher or the same with respect to that of the same object at rest?". We conclude that such a question is ill posed and has no unique answer. We are not claiming a comprehensive review of the subject. In the historical discussion, we give special emphasis to the contribution on the subject by Professor Blanuša from Zagreb.

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1. *Introduction*

Attempts to modify thermodynamics in order to render it compatible with relativity led to long discussions and controversies in the sixties. Relativistic thermodynamics goes back, as a programme, to the early years of relativity (1907), with A. Einstein and M. Planck as initiators. Later in the literature two approaches were used: classical thermodynamics and classical statistical mechanics.

In this paper we want to discuss the relativistic transformation of the familiar and intuitive concept of temperature.

Let T and S be the temperature and the entropy of a body at rest in a reference frame K, and T' and S' be the same quantities measured in a frame K' moving with respect to K with a relativistic velocity $\vec{v}$ (the velocity of light being taken as unity). Then according to Planck and Einstein [1] one has:

$$T' = T\sqrt{1 - v^2}; \quad S' = S. \tag{1}$$

In 1923 A. S. Eddington (on a weaker ground) [2], and in 1947 D. Blanuša (by a more elaborated approach) [3] proposed instead:

$$T' = \frac{T}{\sqrt{1 - v^2}}; \quad S' = S. \tag{2}$$

In 1963, in a very elaborated paper [4], H. Ott reconsidered and criticised the arguments leading to Eqs. (1) and concluded that expression (2) is correct and (1) is wrong. This was the beginning of the discussions and controversies on the matter recalled above.

Later P. T. Landberg [5] stressed that the only adequate transformation should have the form:

$$T' = T, \quad S' = S. \tag{3}$$

All the above-mentioned discussions had a conceptual significance, but no practical application.

Today in relativistic heavy-ion interactions, the dominant models are in the framework of relativistic thermodynamics, and temperature is measured indirectly through transverse momentum distributions.

2. *Some historical considerations*

In this section we would like to shortly report a few forgotten historical facts. As stated, the discussion about proper temperature transformations started only in 1963 with Ott's paper, neglecting the work by Blanuša and his report about a correspondence with W. Pauli and Pauli's student R. Schafroth, who had concluded that both formulae (1) and (2) are acceptable from the point of view of relativistic thermodynamics: which one is to be accepted depends on the definition of the quantities involved. Unfortunately Blanuša did not accept this conclusion, which we think to be correct (see Sect. 3). Even in a recent, not so scientific paper, V. Devidé [6] emphasized the importance of Blanuša's paper (to an extent which we feel is exaggerated), attributing the merit to him of "having found an error in Einstein's and Planck's papers", which the scientific community had instead attributed to Ott.

Let us add that in 1967 C. Møller [7], unaware of Blanuša's work and Pauli's comments, gave credit to Ott to have found the "correct" formula (2). The title of his paper was: "A Strange Incident in the History of Physics". For him, the strange thing was that everybody had accepted a wrong formula for 56 years. As we will discuss in Sect. 3, this is not true.

3. *Lorentz transformation of temperature*

Following an unpublished paper by R. Hagedorn, in this section we try to show explicitly that the three possible formulae for the transformation of temperature are connected with the definition of what one calls "temperature" in relativistic thermodynamics.

Relativity sets up a one-to-one correspondence between points of the four-dimensional continuum in reference systems K and K' that are in relative uniform motion. The point $x_A \equiv (t, x, y, z)$ is said to be identical to the point $x'_A \equiv (t', x', y', z')$ when the four coordinates are connected by Lorentz transformation. Then it is possible to define a tensor for a given physical quantity at the point (t, x, y, z) and take the transformed tensor at the point (t', x', y', z') as the same quantity in K'. Then we can use the powerful tool of covariant tensor calculus. If, in the spirit of relativity, we insist that all physical quantities should be tensors (the word includes scalars and vectors), we can easily avoid the conceptual errors so often done in the literature. A well-known example of such errors is the concept of "relativistic mass increase", that Einstein criticized already in 1948 as conceptually wrong in a letter to L. Barnett [8]. Indeed, in passing from K to K' what changes is not the mass, but the components of the four-vector

$$p^\mu = mu^\mu = m(\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}}) .$$

The fact that in the rest frame only the zero component p_R^0 remains different from zero and reduces to the mass, explains the temptation to call $m/\sqrt{1-v^2}$ "relativistic mass". It is, however, against the spirit of relativity to split a four-vector p^μ into a single-component object $m/\sqrt{1-v^2}$, which is not a scalar, and a four-component object $(1, \vec{v})$, which is not a vector. Obviously, many other such splittings are possible, each leading to another " transformation", e.g.,

$$p^\mu = m\sqrt{1-v^2}(\frac{1}{1-v^2}, \frac{\vec{v}}{1-v^2}) .$$

A similar problem arises if one tries to transform the temperature as a one-component object, as seen in the Introduction.

Let us now illustrate a simple example that shows the power of covariant statistical thermodynamics, through which we outline a possible method of defining the temperature of a body in the systems K and K'. We fix the body to a "black body" in thermal equilibrium with it, and measure the spectrum of the emitted particles: from it we can deduce the temperature of the body via Planck's law. In the body rest system (K) the spectrum is isotropic and follows the law:

$$f(E,T) \sim \frac{1}{e^{E/T} \pm 1}, \quad E = \sqrt{p^2 + m^2} , \tag{4}$$

where E is the particle energy, $\vec{p}$ is the particle momentum, the Boltzmann constant is taken as unity, the plus sign refers to emitted bosons and the minus sign to fermions.

If we can write this distribution in a manifestly covariant way, then we at once get the momentum distribution in any Lorentz frame.

Now, E is the zero component of the emitted particle four-momentum: $p^\mu = (E, \vec{p})$ in the body rest frame. We thus have to construct an invariant quantity from p^μ and something else referring to the temperature, so that such an invariant reduces to E/T in the body rest frame. This "something else" must be a four-vector: let us call it β^μ. Then the invariant is $\beta_\mu p^\mu = E\beta^o - \vec{\beta} \cdot \vec{p}$. In the body rest frame, $\vec{\beta} \cdot \vec{p}$ must vanish and β^o must be equal to $1/T$, hence $\beta^\mu = u^\mu/T$, with $u^\mu = (1/\sqrt{1-v^2}, \vec{v}/\sqrt{1-v^2})$. Thus the manifestly covariant momentum distribution is:

$$f(E,T) \sim \frac{1}{e^{\beta_\mu p^\mu} \pm 1} \tag{5}$$

Expression (5) gives the momentum distribution in any observer's frame (p^μ is the four-momentum of the emitted particle and u^μ is the four-velocity of the emitting source). Thus the simple requirement of an invariant Planck's formula leads us to define the "inverse temperature four-vector" β^μ. However, we could have introduced another four-vector for the temperature, namely, the "temperature four-vector" $T^\mu = Tu^\mu$. In this case the invariant combination reducing to E/T in the rest system reads $(p_\mu T^\mu)/(T_\nu T^\nu)$, and it is easily seen that it coincides with $\beta^\mu p_\mu$.

The expression E/T is uniquely defined in terms of a relativistic invariant: what is not uniquely defined is the concept of "temperature of a moving body". Indeed, one can assume such a temperature to be a scalar, given by $\sqrt{T_\mu T^\mu}$ or $1/\sqrt{\beta_\mu \beta^\mu}$: in this case the transformation law (3) is recovered. But one can also extend to the moving system the association of temperature to the zero component of the four-vector linked to it: by choosing the "temperature four-vector", T^μ one gets the transformation law (2); by choosing the "inverse temperature" four-vector β^μ one gets the transformation law (1). All choices are perfectly legitimate, and, on the ground of relativistic invariance alone, no argument can point to any of them. On the other hand, the choice can be possibly linked to the experimental temperature-measuring procedure. An interesting example has been provided by R. Hagedorn: if the moving body incorporates a thermometer visible from the outside, and its temperature is determined from a photographic image taken as the body is passing by, then the scalar temperature (Eq. (3)) would be obtained. Conversely, if one thinks of deducing the temperature from an energy measurement, presumably one would obtain Eq. (2). Therefore, in any case, the operational details of the measuring procedure should be specified and critically analysed.

In conclusion, we agree with Hagedorn in believing that the question: "How does temperature Lorentz transform?" is ill posed, as there is a priori no unique answer to it. The extensive discussion on the subject, carried out for more than twenty years, seems to have been inconclusive, and could have been avoided if Pauli's statement, previously recalled, had been made known and accepted.

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TRANSFORMACIJE TEMPERATURE U RELATIVISTIČKOJ TERMODINAMICI

Analiziramo problem transformacije temperature s povijesnog stajališta i pokušavamo odgovoriti na pitanje: "Da li je temperatura tijela koje se giba relativističkom brzinom niža, viša ili jednaka onoj u sustavu tijela?" Naš je zaključak da je to pitanje pogrešno postavljeno i nema jedan odgovor. Ne tvrdimo da je naše razmatranje cjelovito. U povijesnom odsječku posebno naglašavamo doprinos Profesora D. Blanuše o toj temi.