Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

CLASSIFICATION OF ALL SINGLE TRAVELING-WAVE SOLUTIONS TO GETMANOU EQUATION

XUE-FEI FAN

Department of Mathematics, Northeast Petroleum University, Daqing 163318, China
and
College Arts and Sciences, Heilongjiang Bayi Agricultural University, Daqing 163319, China

Received 1 July 2010; Accepted 9 February 2011
Online 15 April 2011

Under the travelling-wave transformation, Getmanou equation is reduced to a second-order ordinary differential equation. According to the complete discrimination system for polynomials, we obtain all single travelling-wave solutions to Getmanou equation.

PACS number: 05.45.Yv UDC 532.592, 533.951

Keywords: travelling-wave solutions, complete discrimination system for polynomials, Getmanou equation

1. Introduction

There are many methods to find exact travelling-wave solutions to nonlinear partial differential equations. An important aim is to find all single travelling-wave solutions to some equations. For example, using dynamic system method, all travelling-wave solutions to some equations can be analyzed in detail [1, 2]. On the other hand, in a series papers [3 – 9], Liu introduced the complete discrimination system for polynomials to give the classifications of all single travelling-wave solutions to many nonlinear mathematical physics equations. According to Liu's method and other methods, Wang and Li [10] had further studied travelling-wave solutions to some nonlinear equations, and Yang [11] gives all envelope travelling-wave solutions to DaveyStewartson (DS) equation.

In the present paper, we study the Getmanou equation [12]. Using the complete discrimination system to the fifth-order polynomial, we give directly the classification of all single travelling-wave solutions. It would be difficult to discuss all values of parameters in Getmanou equation in detail.

2. *Classification*

The Getmanou equation reads

$$u_{xt} + \frac{u_x u_t}{1-u^2} - u(1-u^2) = 0\,. \tag{1}$$

Taking the travelling-wave transformation as $u = u(\xi)$, $\xi = kx + \omega t$, Eq. (1) is reduced to the following ODE

$$u'' + \frac{1}{1-u^2}(u')^2 + \frac{u(u^2-1)}{k\omega} = 0\,, \tag{2}$$

which has the general solution given by

$$\pm(\xi-\xi_0) = \int \frac{\mathrm{d}u}{\sqrt{\left|\frac{u-1}{u+1}\right| \left[C - \frac{2}{k\omega}\int u(u^2-1)\left|\frac{u+1}{u-1}\right| \mathrm{d}u\right]}}\,, \tag{3}$$

where C and ξ_0 are two arbitrary constants. We give the classification of all single travelling-wave solutions to equation (1) as follows.

Case 1. If $u > 1$ or $u < -1$, the integral (3) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int \sqrt{\frac{u+1}{-(u+1)^5 + \frac{10}{3}(u+1)^4 - \frac{8}{3}(u+1)^3 + d(u+1) - 2d}}\,\mathrm{d}u\,, \tag{4}$$

where $d = 2k\omega \mathrm{C} - \frac{1}{3}$.

Case 1.1. $d = 0$. Then the corresponding integral (4) is

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int \frac{\mathrm{d}u}{(u+1)\sqrt{(1-u)(u-\frac{1}{3})}}\,. \tag{5}$$

Because the radical has no sense, Getmanou equation has no solution.

Case 1.2. $d \neq 0$. Denote $F(v) = v^5 + pv^3 + qv^2 + rv + s$, where $v = -u - \frac{1}{3}$, $p = -\frac{16}{9}$, $q = \frac{16}{27}$, $r = \frac{16}{27} - d$, $s = -\frac{4}{3}(d + \frac{16}{81})$. The integral (4) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int \sqrt{\frac{\frac{2}{3} - v}{v^5 + pv^3 + qv^2 + rv + s}}\,\mathrm{d}v\,. \tag{6}$$

The complete discrimination system for the polynomial $F(v)$ is given by [6]

$$D_2 = -p\,,$$

$$D_3 = 40rp - 12p^3 - 45q^2,$$

$$D_4 = 12p^4r - 4p^3q^2 + 117pq^2r - 88p^2r^2 - 40p^2qs - 27q^4 - 300qrs + 160r^3,$$

$$D_5 = -1600qr^3s - 3750pqs^3 + 2000pr^2s^2 - 4p^3q^2r^2 + 16p^3q^3s - 900p^3rs^2$$

$$+825p^2q^2s^2 + 144pq^2r^3 + 2250q^2rs^2 + 16p^4r^3 + 108p^5s^2 - 128p^2r^4$$

$$-27q^4r^2 + 108q^5s + 256r^5 + 3125s^4 - 72p^4qrs + 560p^2qr^2s - 630pq^3rs\,,$$

$$E_2 = 160p^3r^2 + 900q^2r^2 - 48p^5r + 60p^2q^2r + 1500pqrs$$

$$+16p^4q^2 - 1100p^3qs + 625p^2s^2 - 3375q^3s\,,$$

$$F_2 = 3q^2 - 8pr\,. \tag{7}$$

Case 1.2.1. $D_5 = 0,\ D_4 > 0$, then $d = -\frac{1}{3}$ or $d = 0$. Therefore we have $F(v) = (v-\alpha)^2(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$, where $\alpha,\ \alpha_1,\ \alpha_2,\ \alpha_3$ are real numbers and $\alpha_1 > \alpha_2 > \alpha_3$. We have

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \frac{2(\frac{2}{3}-\alpha)}{(\alpha_1-\alpha)(\alpha-\alpha_2)\sqrt{(\frac{2}{3}-\alpha_2)(\alpha_1-\alpha_3)}}\,(\alpha_1-\alpha_2)\prod\left(\lambda, \frac{(\frac{2}{3}-\alpha_1)(\alpha-\alpha_2)}{(\frac{2}{3}-\alpha_2)(\alpha-\alpha_1)}, r\right)$$

$$+(\alpha-\alpha_1)F(\lambda, r)] - \frac{2}{\sqrt{(\frac{2}{3}-\alpha_2)(\alpha_1-\alpha_3)}}F(\lambda, r)\,, \tag{8}$$

where $\lambda = \arcsin\sqrt{\frac{(\frac{2}{3}-\alpha_2)(-u-\alpha_1-\frac{1}{3})}{(\frac{2}{3}-\alpha_1)(-u-\alpha_2-\frac{1}{3})}}$, $r = \sqrt{\frac{(\frac{2}{3}-\alpha_1)(\alpha_2-\alpha_3)}{(\frac{2}{3}-\alpha_2)(\alpha_1-\alpha_3)}}$. It is clear that Eq. (8) is elliptic integral of the first kind and the third kind.

Case 1.2.2. $D_5 = 0,\ D_4 < 0$, then $d = \frac{16}{3}$. Therefore we have $F(v) = (v-\alpha)^2(v-\beta)[(v-l)^2+s^2]$, where $\alpha,\ \beta,\ l, s$ are real numbers. The corresponding integral (4) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int\frac{1}{u+\alpha+\frac{1}{3}}\sqrt{\frac{1+u}{-(u+\beta+\frac{1}{3})[(u+l+\frac{1}{3})^2+s^2]}}\,\mathrm{d}u\,. \tag{9}$$

It is clear that the integrals (9) can be expressed by elliptic integrals of the first kind and the third kind.

Case 1.2.3. $D_5 < 0$, then $d > \frac{1}{3}$. Therefore we have $F(v) = (v - \alpha_1)(v - \alpha_2)(v - \alpha_3)[(v-l)^2+s^2]$, where $\alpha_1,\ \alpha_2,\ \alpha_3,\ l, s$ are real numbers. The corresponding integral (4) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}}=\int v\sqrt{\frac{1+u}{-(u+\alpha_1+\frac{1}{3})(u+\alpha_2+\frac{1}{3})(u+\alpha_3+\frac{1}{3})[(u+l+\frac{1}{3})^2+s^2]}}\,\mathrm{d}u\,, \qquad (10)$$

which can be expressed by the hyper-elliptic function or hyper-elliptic integrals.

Case 1.2.4. $D_5 > 0$, $D_3 < 0$, then $d < -\frac{1}{3}$. Therefore we have $F(v) = (v - \alpha)[(v - l_1)^2 + s_1^2][(v - l_2)^2 + s_2^2]$, where $\alpha,\ l_1, l_2, s_1, s_2$ are real numbers. The corresponding integral (4) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}}=\int\sqrt{\frac{1+u}{-(u+\alpha+\frac{1}{3})[(u+l_1+\frac{1}{3})^2+s_1^2][(u+l_2+\frac{1}{3})^2+s_2^2]}}\,\mathrm{d}u\,, \qquad (11)$$

which can be expressed by the hyper-elliptic function or hyper-elliptic integrals.

Case 2. If $-1 < u < 1$, the integral (3) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}}=\int\sqrt{\frac{u+1}{-(u+1)^5+\frac{10}{3}(u+1)^4-\frac{8}{3}(u+1)^3-d(u+1)+2d}}\,\mathrm{d}u\,, \qquad (12)$$

where $d = 2k\omega\mathrm{C}+\frac{1}{3}$.

Case 2.1. $d = 0$. Then the integral (12) is

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}}=\int\frac{\mathrm{d}u}{(u+1)\sqrt{(1-u)(u-\frac{1}{3})}}\,. \qquad (13)$$

When $\frac{1}{3} < u < 1$, the solutions to equation (1) are given by

$$u=\frac{8}{5\pm\sin\dfrac{2(\xi-\xi_0)}{\sqrt{3k\omega}}}-1\,, \qquad (14)$$

which are double-periodic solutions.

Case 2.2. $d\neq 0$. Denote $F(v) = v^5 + pv^3 + qv^2 + rv + s$, where $v = -u - \frac{1}{3}$, $p = -\frac{16}{9},\ q = \frac{16}{27},\ r = \frac{16}{27} + d,\ s = \frac{4}{3}(d - \frac{16}{81})$.

Case 2.2.1. $D_5 = 0,\ D_4 < 0$, then $d = -\frac{16}{3}$ or $d = 0$ or $d = \frac{1}{3}$. Therefore we have $F(v) = (v-\alpha)^2(v-\beta)[(v-l)^2+s^2]$, where $\alpha,\ \beta,\ l, s$ are real numbers. The integral (12) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int \frac{1}{u+\alpha+\frac{1}{3}}\sqrt{\frac{1+u}{-\,(u+\beta+\frac{1}{3})[(u+l+\frac{1}{3})^2+s^2]}}\,\mathrm{d}u\,, \tag{15}$$

which can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Case 2.2.2. $D_5 < 0$, then $d < \frac{1}{3}$. Therefore we have $F(v) = (v-\alpha_1)(v-\alpha_2)(v-\alpha_3)[(v-l)^2+s^2]$, where $\alpha_1,\ \alpha_2,\ \alpha_3,\ l, s$ are real numbers. The integral (12) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int v\sqrt{\frac{1+u}{-(u+\alpha_1+\frac{1}{3})(u+\alpha_2+\frac{1}{3})(u+\alpha_3+\frac{1}{3})[(u+l+\frac{1}{3})^2+s^2]}}\,\mathrm{d}u\,, \tag{16}$$

which can be expressed by the hyper-elliptic function or hyper-elliptic integrals.

Case 2.2.3. $D_5 > 0,\ D_3 < 0$, then $d > \frac{1}{3}$. Therefore we have $F(v) = (v-\alpha)[(v-l_1)^2+s_1^2][(v-l_2)^2+s_2^2]$, where $\alpha,\ l_1, l_2, s_1, s_2$ are real numbers . The integral (12) becomes

$$\pm\frac{\xi-\xi_0}{\sqrt{2k\omega}} = \int \sqrt{\frac{1+u}{-\,(u+\alpha+\frac{1}{3})[(u+l_1+\frac{1}{3})^2+s_1^2][(u+l_2+\frac{1}{3})^2+s_2^2]}}\,\mathrm{d}u\,. \tag{17}$$

We must express the integrals (17) by the hyper-elliptic function or hyper-elliptic integrals.

3. Conclusion

In summary, under the travelling-wave transformation, we reduce Getmanou equation to the corresponding ordinary differential equation (ODE). Using the complete discrimination system for polynomial and the detailed discussion about concrete parameter's values, we obtain its all single travelling-wave solutions. Among these, two solutions are triangle function periodic solutions and others can be expressed by elliptic function and the hyper-elliptic function.

References

[1] Zheng-rong Liu and Ji-bin Li, Int. J. Bifurc. Chaos **12** (2002) 397.

[2] Ji-bin Li, Jian-hong Wu and Huai-ping Zhu, Int. J. Bifurc. Chaos **16** (2006) 2235.

[3] Cheng-Shi Liu, Comput. Phys. Comm. **181** (2010) 317.
[4] Cheng-Shi Liu, Comm. Theor. Phys. **49** (2008) 291.
[5] Cheng-Shi Liu, Comm. Theor. Phys. **49** (2008) 153.
[6] Cheng-Shi Liu, Chin. Phys. **16**, 7 (2007) 1932.
[7] Cheng-Shi Liu, Comm. Theor. Phys. **45**, 6 (2006) 991.
[8] Cheng-Shi Liu, Chin. Phys. **14**, 9 (2005) 1710.
[9] Cheng-Shi Liu, Comm. Theor. Phys. **48**, 4 (2007) 601.
[10] Deng-shan Wang and Hong-bo Li, J. Math. Anal. Appl. **343**, 1 (2008) 273.
[11] Shu Yang, Mod. Phys. Lett. B **24** (2010) 363.
[12] Sen-Yue Lou, Phys. Lett. A **187** (1994) 239.

RAZVRSTAVANJE SVIH JEDNOVALNIH RJEŠENJA GETMANOUOVE JEDNADŽBE

Primjenom pretvorbe za putujući val, Getmanouova se jednadžba svodi na diferencijalnu jednadžbu drugog reda. Pomoću potpunog razlikovnog sustava za polinome, izvodimo sva rješenja te jednadžbe za jednovalne putujuće valove.