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GENERALIZED FRACTAL SPACE FROM ERDÉLYI-KOBER FRACTIONAL INTEGRAL

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It is argued that the distribution on fractal with fractional mass dimension and integer mass dimension can be described simultaneously by means of the Erdélyi-Kober fractional integral.

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The fractional calculus is the field of applied mathematics which deals with the application of integrals and derivatives of arbitrary order [1 − 5]. It is an old topic since Leibniz (1695, 1697) and Euler (1730) ideas of generalizing the notion of derivative to non-integer order, in particular to the order 1/2. In recent years, considerable interest has been stimulated by many applications in different fields of sciences, ranging from the theory of analytic functions to statistical physics, quantum mechanics, astrophysics, control theory, and quantum field theory [6 − 38]. In fact, most of the mathematical theories applicable to the study of derivatives and integrals of noninteger order were developed prior to the turn of the 20th century, in particular, numerous applications and physical manifestations of fractional calculus have been found.

Although the fractional theory is very rich, it was considered for more than three centuries as a theoretical mathematical field with no physical interests. During the last decade, the pure mathematics has changed in some particular cases to meet the requirements of physical reality and objectives. However, the mathematical background surrounding fractional calculus is far from paradoxical. While the physical meaning is difficult to grasp, the fractional operators themselves are no more rigorous than those of their integer order counterparts. Today there exist many different approaches to the fractional calculus, not all being equivalent, ranging from divided-difference types to infinite-sum types, including the Grunwald-Letnikov fractional derivative, the Caputo fractional derivative, the Erdélyi-Kober fractional derivative, and so on. That is the concept of differ-integral of complex order can be introduced in several ways. Among mathematicians, the most widely used definition of an integral of fractional integral is via an integral transform, known as the Riemann-Liouville fractional integral (RLFI) and is defined as follows

$$
\begin{aligned}
{}_aI_x^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-X)^{\alpha-1} f(X)\,\mathrm{d}X, \quad \text{for} \quad \Re e(\alpha) > 0\,, \\
&= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\ {}_aI_x^{\alpha+n} f(x), \quad \text{for} \quad -n < \Re e(\alpha) \le 0\,.
\end{aligned} \tag{1}
$$

An operator of fractional integration generalizing the RLFI through association of power weight was introduced by Erdélyi and Kober (1940), Kober (1940), Erdélyi (1951) as follows [39],

$$
\begin{aligned}
{}_aI_x^{\eta,\alpha} f(x) &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int\limits_a^x (x-X)^{\alpha-1} X^{\eta} f(X)\,\mathrm{d}X, \quad \text{for} \quad \Re e(\alpha) > 0\,, \\
&= x^{-\eta-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}x^n} x^{\eta+\alpha+n}\ {}_aI_x^{\eta,\alpha+n} f(x), \quad \text{for} \quad -n < \Re e(\alpha) \le 0\,,
\end{aligned} \tag{2}
$$

where $f(X) \in L_p(0,\infty)$. Despite the fact that the Erdélyi-Kober fractional differentiation operator is a very general object containing many operators of fractional calculus as its particular cases, a number of different modifications of them have been studied by many authors [40−45]. All known operational analysis for this kind of operators were constructed for some particular cases of the composition of the left-hand sided (or right-hand sided) Erdélyi-Kober operators. A generalization of the left- and right-hand sided Erdélyi-Kober fractional differentiation operators are defined as follows.

Given a function $f \in L_p(0,\infty)$, the generalized left- and right-hand sided Erdélyi-Kober fractional integration operators (EKFI) are defined in our framework by:

$$
I^{\eta,\alpha} f(X) = \frac{2X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int\limits_0^X (X^2 - x^2)^{\alpha-1} x^{2\eta+1} f(x)\mathrm{d}x\,, \tag{3}
$$

$$
J^{\eta,\alpha} f(X) = \frac{2X^{2\eta}}{\Gamma(\alpha)} \int\limits_X^\infty (x^2 - X^2)^{\alpha-1} x^{-2(\eta+\alpha)+1} f(x)\mathrm{d}x\,. \tag{4}
$$

if $\Re e(\alpha) \ge 0$ and $\Re e(\eta) > -1/2$. These fractional integrals have been used by several authors, in particular, to get solutions of the single, dual and triple integral equations possessing particular functions of mathematical physics as their kernels. It was recently pointed using the dimensional regularizations technique that the interpretation of Riemann-Liouville fractional integration is connected with fractional dimension. In this framework, the fractal-like mass in a box of size X in a region $W = [x : |x| \le X]$ obeys a power-law relation $M(X) \propto X^D$ where D is the mass dimension of the medium [46]. As the definition (3) is a very general form, the main focus in this work is to demonstrate how the EKFI can be used to obtain a better generalized form for $M(X)$. We can realize this objective by the fractional generalization of the equation

$$M(X) = \int_0^X \rho(x)\mathrm{d}x\,, \tag{5}$$

in the line $0 < x < X$, where $\rho(x) \in L_1(R^1)$.

We define the EKFI of Eq. (5) by

$$M^{\eta,\alpha}(X) = \frac{2X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^X (X^2 - x^2)^{\alpha-1} x^{2\eta+1} \rho(x)\mathrm{d}x\,. \tag{6}$$

For the case of a fractional continuous medium model with density distribution $\rho(x) = \rho_0 x^{2r}, r \in \mathbb{R}, \rho_0 = constant$, Eq. (6) gives

$$M^{\eta,\alpha}(X) = \rho_0 \frac{\Gamma(\eta + 1 + r)}{\Gamma(\alpha + \eta + 1 + r)} X^{2r}. \tag{7}$$

It follows that $D = 2r$, i.e. the fractal mass dimension depends on the exponent of the density distribution. Amazingly, for a constant density distribution, i.e. $r = 0$, $M^{\eta,\alpha}(X)$ is independent of X and hence we argue that the EKFI does not hold for fractional continuous models with constant energy densities. For $r = 3/2$, we obtain straightforwardly

$$M^{\eta,\alpha}(X) = \rho_0 \frac{\Gamma(\eta + 5/2)}{\Gamma(\alpha + \eta + 5/2)} X^3. \tag{8}$$

Another interesting example concerns the case where $\rho(x) = \rho_0 x^{-2r}(X + x)^{1-\alpha}$. We find straightforwardly

$$M^{\eta,\alpha}(X) = \frac{2\rho_0 X^{-2(\eta+\alpha)}}{\Gamma} \int_0^X (X - x)^{\alpha-1} X^{2\eta+1-2r}\mathrm{d}x \tag{9}$$

$$= 2\rho_0 \frac{\Gamma(2\eta + 2 - 2r)}{\Gamma(2\eta + 2 - 2r + \alpha)} X^{1-2r-\alpha}, \quad \Re e(2\eta + 1 - 2r) > 0, \quad \Re e(\alpha) > 0. \tag{10}$$

One more interesting example concerns the case where $\rho(x) = \rho_0(X^2 - x^2)^{1-\alpha}$. Consequently, we find straightforwardly

$$M^{\eta,\alpha}(X) = \frac{\rho_0}{(\eta+1)\Gamma(\alpha)} X^{2(1-\alpha)}. \tag{11}$$

A further illustration concerns the case where $\rho(x) = \rho_0 x^{-(1+2\eta)}(X+x)^{1-\alpha}\log x$. Accordingly, we get

$$M^{\eta,\alpha}(X) = \frac{2\rho_0 X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int\limits_0^X (X-x)^{\alpha-1}\log x \,\mathrm{d}x \tag{12}$$

$$= \frac{2\rho_0 X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \left\{ X^\alpha \log X \int\limits_0^1 Y^{\alpha-1}\mathrm{d}Y + X^\alpha \int\limits_0^1 Y^{\alpha-1}\log(1-Y)\mathrm{d}Y \right\}, \quad x = X(1-Y)\,. \tag{13}$$

After integrating by parts and making use of the fact that [47]

$$\int\limits_0^1 \frac{Y^m - Y^n}{1-Y}\,\mathrm{d}Y = \Psi(n+1) - \Psi(m+1), \quad \Re e(m), \Re e(n) > -1\,, \tag{14}$$

where

$$\Psi(n) \triangleq \frac{1}{\Gamma(n)}\,\frac{\mathrm{d}}{\mathrm{d}n}\Gamma(n) \tag{15}$$

is the psi function obeying the recursion relation

$$\Psi(n+1) - \Psi(n) = \frac{1}{n}\,, \tag{16}$$

with $\Psi(1) = -0.5772157\ldots \equiv \gamma$ (the Euler number), we find:

$$M^{\eta,\alpha}(X) = \frac{2\rho_0 X^{-2\eta-\alpha}}{\Gamma(\alpha+1)} \{\log X - \Psi(\alpha+1) + \Psi(1)\}\,. \tag{17}$$

For the case $\rho(x) = \rho_0 x^{-1-2\eta}[1 + (X+x)^{1-\alpha}\log x]$, we obtain

$$M^{\eta,\alpha}(X) = \frac{2\rho_0 X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \int\limits_0^X (X^2 - x^2)^{\alpha-1}\{1 + (X+x)^{1-\alpha}\log x\}\mathrm{d}x \tag{18}$$

$$= \frac{2\rho_0 X^{-2(\eta+\alpha)}}{\Gamma(\alpha)} \left[\int\limits_0^X (X^2 - x^2)^{\alpha-1}\mathrm{d}x + \int\limits_0^X (X-x)^{\alpha-1}\log x\,\mathrm{d}x \right] \tag{19}$$

$$=2\rho_0 X^{-2(\eta+1)}\left[\frac{\sqrt{\pi}}{2\Gamma(\alpha+1/2)}+\frac{1}{\Gamma(\alpha+1)}X^{1-2\alpha}\{\log X-\Psi(\alpha+1)+\Psi(1)\}\right]. \quad (20)$$

For $\alpha = -\eta$, i.e. $\rho(x) = \rho_0 x^{-1+2\alpha}[1 + (X + x)^{1-\alpha} \log x]$, Eq. (20) gives

$$M^{\eta,\alpha}(X)=\frac{2\rho_0}{\Gamma(\alpha+1)}\{\Psi(1)-\Psi(\alpha+1)\}+\rho_0\frac{\sqrt{\pi}}{\Gamma(\alpha+1/2)}X^{2\alpha-1}+\frac{2\rho_0}{\Gamma(\alpha+1)}\log X. \quad (21)$$

Last term in Eq. (21) is the logarithmic correction.

These illustrations may be helpful to understand many aspects of dark matter halo of disc galaxies and rotation curve of a dark matter filament [48] as well as many black holes logarithmic corrections without passing through the details of quantum gravity [49, 50].

The three-dimensional fractional generalization of the equation

$$M_3(X) = \iiint\limits_W \rho(x)\ \mathrm{d}^3x \quad (22)$$

is derived as follows: we define the generalized EKFI of Eq. (5) for $\rho(x) = \rho_0 x^{2r}$ by

$$M^{\eta,\alpha}(X)=\frac{2^3\prod_{i=1}^3 X_i^{-2(\eta_i+\alpha_i)}}{\prod_{i=1}^3\Gamma(\alpha_i)}\iiint\limits_W\left\{\prod_{i=1}^3\left(X_i^2-x_i^2\right)^{\alpha_i-1}x_i^{2\eta_i+1}\right\}\rho(x)\ \mathrm{d}^3x \quad (23)$$

$$=\rho_0\frac{2^3\prod_{i=1}^3 X_i^{-2(\eta_i+\alpha_i)}}{\prod_{i=1}^3\Gamma(\alpha_i)}\int\limits_0^{X_3}\int\limits_0^{X_2}\int\limits_0^{X_1}(X_3^2-x_3^2)^{\alpha_3-1}(X_2^2-x_2^2)^{\alpha_2-1}(X_1^2-x_1^2)^{\alpha_1-1}$$

$$\times\{x_3^{2\eta_3+1+2r_1}x_2^{2\eta_2+1+2r_2}x_1^{2\eta_1+1+2r_3}\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1\} \quad (24)$$

$$=2^3\rho_0\frac{\prod_{i=1}^3\Gamma(\eta_i+1+r_i)}{\prod_{i=1}^3\Gamma(\alpha_i+\eta_i+1+r_i)}\prod_{i=1}^3 X_i^{2r_i}. \quad (25)$$

If for instance, $X_1 = X_2 = X_3 = X$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ and $\eta_1 = \eta_2 = \eta_3 = \eta$, then

$$M^{\eta,\alpha}(X) = 2^3\frac{\Gamma(\eta+1+r)}{\Gamma(\alpha+\eta+1+r)}\,. \quad (26)$$

In summary, we have then a generalization of the standard case. These results could not be found using the Riemann-Liouville fractional integral and could be practical to understand many aspects in astrophysics. It is noteworthy that to better quantify the self-similar structure in astrophysics, a number of works have revealed that the interstellar clouds obey certain power-law relations between size and mass [51]. These scaling relations are observed whatever the tracer, and the size-mass relation that follows is $M \propto R^D$ with D the Hausdorff fractal dimension between 1.6 and 2. Recently, Heithausen et al. [52] extended the size-line-width and size-mass relations down to Jupiter masses; their mass-size relation is $M \propto R^{2.31}$, much steeper than earlier studies, but the evaluation of masses at small scales is moderately doubtful [52, 53]. It appears also that the self-similarity of structures is broken in regions of star formation. We conclude that the benefit of using EKFI is clear and besides that the notion of Erdélyi-Kober fractional integral may be connected with both fractional and integer dimensions simultaneously. Further consequences are under progress, in particular the Erdélyi-Kober fractional measure aspects.

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POOPĆEN RAZLOMNI PROSTOR IZ ERDÉLYI-KOBEROVOG RAZLOMNOG INTEGRALA

Raspravljamo kako se raspodjela razlomnika s razlomnom i cjelobrojnom dimenzijom mase mogu istodobno opisati pomoću Erdélyi-Koberovog razlomnog integrala.