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# COORDINATE TIME HYPERBOLIC MOTIOM FOR BELL'S SPACESHIP EXPERIMENT

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A simple and comprehensive solution is given for the Bell thought experiment. Using the equations of hyperbolic motion expressed in coordinate time, we can easily derive the distance between rockets as a function of time, the time elapsed between rocket takeoff and string breaking as well as the expression that gives the exact coordinate time of string breaking.

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## 1. *Introduction*

In Bell's "spaceship" experiment [1], two spaceships, that are initially at rest in some common inertial reference frame, are connected by a taut string. At the time zero in the common inertial frame, both spaceships start accelerating, with a constant proper acceleration a as measured by an on-board accelerometer. We study the question: when does the string break as expressed as a function of the coordinate time? For simplicity, throughout the paper, all objects (string, rockets) are considered as being Born-rigid [2], thus neglecting the very minor effects on the length of the objects during the accelerated motion [3 − 7]. We have provided earlier [9] a treatment expressed in terms of the proper time, $\tau$, while in the current paper we show a different approach in terms of the coordinate time, $t$.

## 2. *Analysis of motion assuming hyperbolic dependence of position as function of coordinate time*

Bell's paradox is easily understood if we start by looking at the situation in the instantaneously co-moving inertial rest frame (ICIRF) of the rear spaceship at a given instant of time. In that frame and at that time, the trailing spaceship is at rest, and at the same time the leading spaceship has nonzero velocity moving forward.

We do this for the ICIRF of any small portion of the string, and each spaceship is moving away from the portion in question. So from the standpoint of the string, each small region of the string must stretch and eventually break when its elastic limit is exceeded. In Ref. [9], we have shown a treatment expressed in terms of the proper time, $\tau$. In the current paper, we show a different treatment, expressed in terms of the coordinate time $t$. The two treatments are different both physically and mathematically.

Two rockets, A and B, describe hyperbolic trajectories [9] (see Fig. 1),

$$x_{\text{leading}} = L + \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2}, \qquad x_{\text{trailing}} = \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2}. \tag{1}$$

Let us consider the instantaneous co-moving frame $\Sigma$ (an inertial frame with the origin attached to the trailing rocket) at an arbitrary coordinate time $t_A$. The $\xi$ axis makes an angle $\alpha$ with the $x$ axis, where

$$\tan\alpha = \frac{v(t_A)}{c} = \frac{\dfrac{at_A}{c}}{\sqrt{1+\left(\dfrac{at_A}{c}\right)^2}}\,. \tag{2}$$

The $\xi$ axis represents a line of simultaneity in the frame $\Sigma$. If we want to determine the distance between the two rockets at the coordinate time $t_A$ as measured in $\Sigma$,

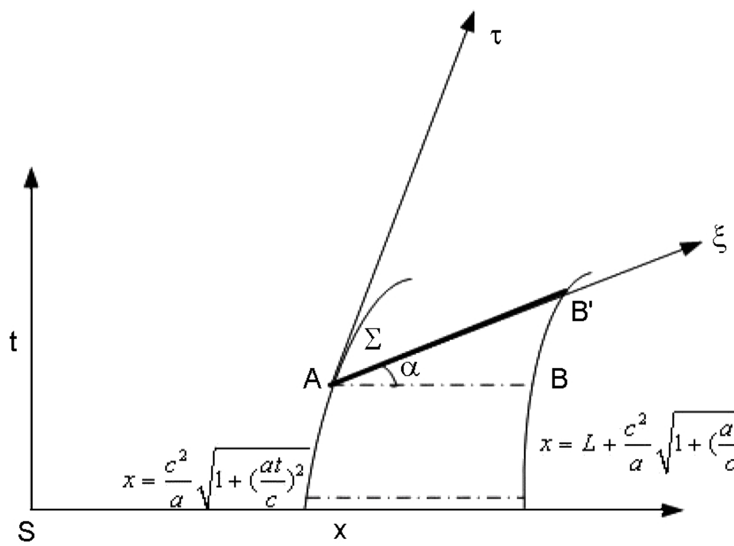


*Fig. 1. Minkowski's diagram for the Bell's spaceship experiment.*

we have to intersect the $\xi$ axis with the trajectory of the leading rocket, that is, we have to solve for $t$ the system of equations:

$$(x - x_A) \tan\alpha = c(t - t_A)\,, \qquad x = L + \frac{c^2}{a}\sqrt{1 + \left(\frac{at}{c}\right)^2}\,, \tag{3}$$

where

$$x_A = \frac{c^2}{a}\sqrt{1 + \left(\frac{at_A}{c}\right)^2}\,. \tag{4}$$

The system reduces to a simple equation of the second degree in $t$ that has a positive root $t_{B'} > t_A$,

$$\sqrt{1 + \left(\frac{at}{c}\right)^2} = \frac{t}{t_A}\sqrt{1 + \left(\frac{at_A}{c}\right)^2} - \frac{La}{c^2}\,, \tag{5}$$

with the solution:

$$t_{B'} = t_A \frac{La}{c^2}\left(\sqrt{1 + \left(\frac{at_A}{c}\right)^2} + \sqrt{\left(\frac{c^2}{La}\right)^2 + \left(\frac{at_A}{c}\right)^2}\right). \tag{6}$$

From (6) it is obvious that $t_{B'} > t_A$. Once we find $t_{B'}$, we can easily find the coordinate (in frame S) of the leading rocket

$$x_{B'} = L + \frac{c^2}{a}\sqrt{1 + \left(\frac{at_{B'}}{c}\right)^2}\,. \tag{7}$$

We can now apply a Lorentz transformation between the launcher frame S and the co-moving frame $\Sigma$ in order to get the distance between the rockets calculated in the frame $\Sigma$,

$$L' = \xi_{B'} = \gamma(v(t_A))\,(x_{B'} - vt_{B'})\,, \tag{8}$$

where

$$\gamma(v(t_A)) = \sqrt{1 + \left(\frac{at_A}{c}\right)^2}\,, \tag{9}$$

and

$$x_{B'} - vt_{B'} = L + \frac{c^2}{a}\sqrt{1 + \left(\frac{at_{B'}}{c}\right)^2} - \frac{at_A t_{B'}}{\sqrt{1 + \left(\frac{at_A}{c}\right)^2}}\,. \tag{10}$$

We can show easily that the function

$$f(t_{B'}) = \frac{c^2}{a}\sqrt{1 + \left(\frac{at_{B'}}{c}\right)^2} - \frac{at_A t_{B'}}{\sqrt{1 + \left(\frac{at_A}{c}\right)^2}} > 0\,, \tag{11}$$

since

$$\frac{\mathrm{d}f}{\mathrm{d}t_{B'}} = a\left(\frac{t_{B'}}{\sqrt{1+\left(\frac{at_{B'}}{c}\right)^2}} - \frac{t_A}{\sqrt{1+\left(\frac{at_A}{c}\right)^2}}\right) > 0\,, \tag{12}$$

for any $t_{B'} > t_A$. This is due to the fact that $g(t) = t \Big/ \sqrt{1+\left(\frac{at}{c}\right)^2}$ is a monotonically increasing function.

Thus, we can conclude that the distance between the rockets in the co-moving frame is larger than their distance $L$ as measured in the launcher frame

$$\xi_{B'} = \sqrt{1+\left(\frac{at_A}{c}\right)^2}\left(L + \frac{c^2}{a}\sqrt{1+\left(\frac{at_{B'}}{c}\right)^2}\right) - at_A t_{B'} > L\,. \tag{13}$$

Since $t_A$ has been chosen arbitrarily and since $t_{B'}$ is a function of $T(t)$ as described by Eq. (5), we can also write the distance between the two rockets as a function of the coordinate time $t$, as measured in the co-moving frame $\Sigma$, by a more general formula

$$L'(t) = \sqrt{1+\left(\frac{at}{c}\right)^2}\left(L + \frac{c^2}{a}\sqrt{1+\left(\frac{aT(t)}{c}\right)^2}\right) - atT(t)\,, \tag{14}$$

The distance between rockets increases with the coordinate time $t$ so the string will get stretched until it breaks.

## 3. *When does the string break?*

The calculation of the time when the string breaks requires that we take into consideration that each infinitesimal element of the string moves at a different speed as viewed from the frame $\Sigma$, as we have shown in the previous section. So, each infinitesimal element will stretch by a different amount [8]. The formalism built in the prior section will be very useful in calculating the amount of stretching.

Consider an infinitesimal element of length $\mathrm{d}l$ as viewed from the frame. Its endpoints will describe the hyperbolas (see Fig. 2)

$$x_i = \lambda + \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2}\,, \qquad x_{i+1} = \lambda + \mathrm{d}\lambda + \frac{c^2}{a}\sqrt{1+\left(\frac{at}{c}\right)^2}\,. \tag{15}$$

According to (6), the $\xi$ axis intersects the two hyperbolas at

$$t_i = \lambda p\,, \quad t_{i+1} = (\lambda + \mathrm{d}\lambda)p\,, \quad \text{with } p = \frac{at_A}{c^2}\left(\sqrt{1+\left(\frac{at_A}{c}\right)^2} + \sqrt{\left(\frac{c^2}{La}\right)^2 + \left(\frac{at_A}{c}\right)^2}\right). \tag{16}$$

Substituting (16) into (15), we obtain

$$x_{i+1} - x_i = \mathrm{d}\lambda + \frac{c^2}{a}\left(\sqrt{1+\left(\frac{at_{i+1}}{c}\right)^2} - \sqrt{1+\left(\frac{at_i}{c}\right)^2}\right). \tag{17}$$

So, the infinitesimal element is stretched by the amount

$$\frac{c^2}{a}\left(\sqrt{1+\left(\frac{at_{i+1}}{c}\right)^2} - \sqrt{1+\left(\frac{at_i}{c}\right)^2}\right) \approx \frac{a}{2}(t_{i+1}^2 - t_i^2). \tag{18}$$

Substituting (16) into (18) we obtain

$$\frac{a}{2}(t_{i+1}^2 - t_i^2) = \frac{ap^2}{2}((\lambda + \mathrm{d}\lambda)^2 - \lambda^2) \approx ap^2\lambda\mathrm{d}\lambda\,, \tag{19}$$

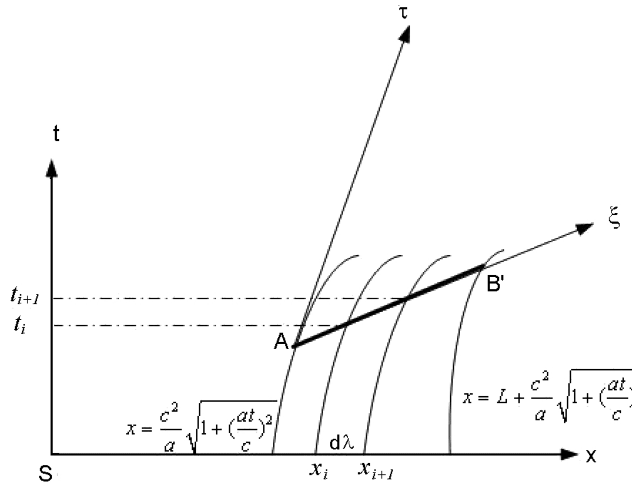


*Fig. 2. Calculation of the stress.*

The total stretch is obtained by integrating (19)

$$\int_0^L ap^2\lambda\mathrm{d}\lambda = \frac{ap^2L^2}{2}. \tag{20}$$

Since $t_A$ has been taken arbitrarily, it means that, in general, $p$ is a function of $t$

$$p = \frac{at}{c^2}\left(\sqrt{1+\left(\frac{at}{c}\right)^2} + \sqrt{\left(\frac{c^2}{La}\right)^2 + \left(\frac{at}{c}\right)^2}\right). \tag{21}$$

Material science teaches us that for any string of a given length $L$, cross-sectional area $A$ and a given tensile strength, there is a limit of stretching, $\delta L$, beyond which it will break. The time of the string breaking will be given by solving the equation

$$\frac{ap^2}{L} = \delta L\,, \tag{22}$$

for the coordinate time $t$.

We have shown the realistic computations of the string stretching as a function of the coordinate time.

## 4. Conclusions

We have produced a very simple and comprehensive solution for the Bell's thought experiment. As we have demonstrated, using the equations of hyperbolic motion, we can easily derive the distance between the two rockets as a function of time as well of the time elapsed between rocket takeoff and string breaking, and the expression that permits the calculation of the exact time of string breaking.

References

[1] J. S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, Cambridge (1987).

[2] E. Dewan and M. Beran, *Note on Stress Effects due to Relativistic Contraction*, Am. J. Phys. **27** (7) (1959) 576.

[3] P. Nawrocki, *Stress Effects due to Relativistic Contraction*, Am. J. Phys. **30** (10) (1962) 383.

[4] E. Dewan and M. Beran, *Stress Effects due to Lorentz Contraction*, Am. J. Phys. **31** (5) (1963) 3.

[5] T. Matsuda and A. Kinoshita, *A Paradox of Two Space Ships in Special Relativity*, AAPPS Bulletin Feb. (2004).

[6] H. Nikolić, *Relativistic Contraction of an Accelerated Rod*, Am. J. Phys. **67** (11) (1999) 1007.

[7] M. Born, *Einstein's Theory of Relativity*, Dover Publications, New York (1962).

[8] O. Grøn, *Covariant formulation of Hooke's law*, Am. J. Phys. **49** (1) (1981) 28.

[9] A. Sfarti, *Hyperbolic motion treatment for Bell's spaceship experiment*, Fizika A **18** (2009) 45.

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Daje se jednostavno i jasno rješenje za Bellov misaoni eksperiment. Primjenom jednadžbi hiperbolnog givanja izraženih u kordinatnom vremenu, jednostavno se izvodi razmak dviju raketa kao funkcija vremena, vrijeme koje protekne od polaska rakete do trenutka pucanja užeta, te se daje izraz za točno koordinatno vrijeme pucanja užeta.