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# TWO-STEP LORENTZ TRANSFORMATION OF FORCE

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Lorentz transformation of force is derived in a more natural way using two boost transformations and corresponding Thomas-Wigner rotation. Some consequences of this transformation are discussed.

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## 1. Introduction

Classical electrodynamics is based on four Maxwell equations and Lorentz force equation. These five equations imply constancy of the speed of light in all reference frames, which is one of the basic assumptions of special theory of relativity. The sixth equation, conservation of electric current, can be understood as a consequence of the Maxwell equations.

In literature one can find many derivations of a subset of these five equations from the special theory of relativity and some additional assumptions.

In the Frisch and Wilets paper [1], Lorentz force law and the Maxwell equations are derived from Lorentz transformations, Gauss's law on the flux of the electric field, and two additional postulates on electromagnetic force. Their derivation is complete and relatively simple, but the basic expression for the force transformation comprises quantities defined in both the initial and the final frame, which is somewhat unexpected.

In the Berkeley Physics Course [2], the magnetic field is derived basically under the same assumptions: Gauss's law, special theory of relativity, and the existence of a frame in which the force does not depend on the test-particle velocity (velocity of the particle on which the force acts). The derivation has been done only for very special choices of velocities of force sources and test particles.

The kinematic analysis of the Lorentz transformation of a force (Lorentz force transformation) is often met in the standard textbooks like Møller [3] or Goldstein [4]. In these textbooks, the starting point is always the Newton's second law of motion, and force is treated like a three-vector object (see Eq. (20)). The obtained results therefore hold generally, and can be applied on every force, including also the Lorentz force.

The derivation of the Lorentz force transformation using the Lorentz transformations of electromagnetic field is given by Jefimenko [5].

In the Feynman Lectures on Physics [6], the Lorentz transformation properties of time and space coordinates are used to derive the expressions for transformations of the scalar and vector potential. Using the definition of electric field in terms of scalar and vector potential, it is shown that the expression for the force of a system of electric charges in motion contains the "magnetic term", $q(\mathbf{v}/\mathbf{c}) \times \mathbf{B}$.

The intention of this paper is to show that, starting from the simple force transformation law given in the Berkeley Physics Course and the notion of the Thomas-Wigner rotation [7, 8], one can derive the general force transformation as in the Frisch and Wilets paper. Although this derivation is longer and more complicated, we find it to be more intuitive. Especially, it shows how the interplay between various representations of the Lorentz group can be used, and the role of the Thomas-Wigner rotation when combining two non-parallel boosts.

In the second section we expose the problem of derivation of Lorentz force from the Gauss's law and special theory of relativity as it was treated in Refs. [1] and [2]. In the third section we present our derivation of the Frisch and Wilets formula for the force transformation. Fourth section compares the two derivations of the Lorentz force transformation and discusses some of the consequences.

## *2. Earlier derivations of the Lorentz transformations of electric and magnetic fields*

Frisch and Wilets [1] started from the kinematic definition of the general force,

$$\mathbf{F} \;=\; \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{1}$$

used the Møller's expression for Lorentz transformation of the force [3], and rearranged it to obtain the specific dependence on the test-particle velocity,

$$\mathbf{F}' \;=\; \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{F}) + \gamma\left[\hat{\boldsymbol{\beta}} \times (\mathbf{F} \times \hat{\boldsymbol{\beta}}) + \boldsymbol{\beta}_2' \times (\mathbf{F} \times \boldsymbol{\beta})\right], \tag{2}$$

where $\mathbf{a} \times \mathbf{b}$ is the cross product of two vectors and $\mathbf{ab}$ their scalar product. $\mathbf{F}'$ and $\mathbf{F}$ are the forces on the test particle in the $S'$ and $S$ frames, respectively, with $S$ and

$S'$ being two arbitrary inertial frames. Moreover, $\boldsymbol{\beta} = \mathbf{v}/c$, $\boldsymbol{\beta}'_{\mathbf{2}} = \mathbf{v}'_{\mathbf{2}}/c$, $\mathbf{v}$ is velocity of the frame $S'$ relative to the $S$ frame, $\mathbf{v}'_{\mathbf{2}}$ is velocity of the test particle in the $S'$ frame, and $\gamma = (1-\beta^2)^{-\frac{1}{2}}$. Hatted vectors denote unit vectors or directions of the coresponding vectors. Frisch and Wilets argumented that if there is a reference frame in which the force does not depend on the velocity of the test particle, the force in any other reference frame can be written as

$$\mathbf{F} \quad = \quad \mathbf{e} + \boldsymbol{\beta}_{\mathbf{2}} \times \mathbf{b}, \tag{3}$$

where both $\mathbf{e}$ and $\mathbf{b}$ are independent of the test-particle velocity. That completes their derivation of the Lorentz force from the Gauss's law and the special theory of relativity. Further, comparing this expression with the force in the frame $S'$,

$$\mathbf{F}' \quad = \quad \mathbf{e}' + \boldsymbol{\beta}'_{\mathbf{2}} \times \mathbf{b}', \tag{4}$$

one can obtain the transformation for the $\mathbf{e}$ and $\mathbf{b}$ components of the force

$$\mathbf{e}' \quad = \quad \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{e}) + \gamma\left[\hat{\boldsymbol{\beta}} \times (\mathbf{e} \times \hat{\boldsymbol{\beta}}) + \boldsymbol{\beta} \times \mathbf{b}\right], \tag{5}$$

$$\mathbf{b}' \quad = \quad \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{b}) + \gamma\left[\hat{\boldsymbol{\beta}} \times (\mathbf{b} \times \hat{\boldsymbol{\beta}}) - \boldsymbol{\beta} \times \mathbf{e}\right]. \tag{6}$$

In order to derive the Maxwell equations for general motion of source and test particle, Frisch and Wilets had to pose four postulates:

1) The invariance of charge expressed as a requirement that the Gauss's law is valid in all inertial frames. Notice that this assumption is one of the Maxwell equations.

2) The electric field $\mathbf{E}$ is independent of the test-particle velocity. Accepting postulate 1., postulate 2. is equivalent to the Lorentz force law.

3) The speed of light being finite, the electric and magnetic fields at time $t$ are determined by the behaviour of the point source(s) at time $t - r/c$.

4) The electric field does depend only on first and second derivatives of the source position.

The existence of the magnetic field is a simple consequence of the first and second postulate. In addition, (5) and (6) present the Lorentz transformations of the electric and magnetic fields. In fact, Frisch and Wilets do not derive Maxwell equations directly, but they construct the expressions for a point-source electric and magnetic field from the above postulates, and those were known to be the only ones which Maxwell equations do admit [9].

In his book [10], Jackson stresses that special theory of relativity and the Lorentz force transformation (2) are not sufficient to deduce the existence of the magnetic field without additional assumptions. He cites Frisch and Wilets paper as a reference in which Maxwell equations were correctly derived.

First two of the above postulates are used in the Purcell's book [2] to derive magnetic field from electric field for some special directions of velocities of the sources of the electric field and of the test particle. The Lorentz force

$$\mathbf{F} \quad = \quad q(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \tag{7}$$

is not derived but assumed, although the second postulate is equivalent to the Lorentz force law, and was used to derive the expression for the magnetic field. In the derivation, the special transformations of the force and electric field are being used. The force boost transformation from the inertial frame in which the test particle is momentary at rest, $S_{\mathrm{R}}$, to the arbitrary inertial frame, $S'$, reads

$$\mathbf{F}' \quad = \quad \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{F}_{\mathrm{R}}) + \frac{1}{\gamma}\Big[\mathbf{F}_{\mathrm{R}} - \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{F}_{\mathrm{R}})\Big] \ , \tag{8}$$

where the notation is the same as in (2), with $\mathbf{F}_{\mathrm{R}}$ corresponding to $\mathbf{F}$ in the test-particle rest frame. The inverse force boost transformation, that is the force boost transformation from $S'$ frame to the $S_{\mathrm{R}}$ frame, is obtained replacing $1/\gamma \to \gamma$ in (8). Notice that, if in (2) one puts $\boldsymbol{\beta}'_2 = -\boldsymbol{\beta}$, Eq. (8) is reproduced. Purcell also used the transformation of the electric field from the frame in which all sources are at rest to an arbitrary Lorentz frame. The transformation of the electric field is performed for a pair of uniformly charged parallel planes with opposite surface charge densities, moving parallel or perpendicularly to the electric field produced by the planes. The results obtained are assumed to be valid for any set of charges,

$$\mathbf{E}' \quad = \quad \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{E}) + \gamma\Big[\mathbf{E} - \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{E})\Big] \ . \tag{9}$$

Here $\mathbf{E}$ is the electric field in the rest frame of all sources, and $\mathbf{E}'$ is the corresponding electric field in an arbitrary inertial frame. A magnetic field contribution does not appear since in the initial frame all sources are assumed to be at rest. Using these two transformations, and very specific configuration of the sources corresponding to a simplified model of a neutral current in the laboratory frame, Purcell derived an expression for the magnetic field which corresponds to the third term in (6). This term was derived for the test-particle velocity direction, $\hat{\boldsymbol{\beta}}$, parallel to the neutral current. Purcell generalizes this result without checking it for general velocity directions of the sources and the test particle.

Generally, there is no frame in which all sources are at rest [10]. Therefore, we will not use (9) in our discussion on electric and magnetic field transformations. We will keep only the force transformation (8) which is simple and quite intuitive. Using a combination of two such Lorentz transformations of force we will derive the Frisch and Wilets expression for the general force transformation.

## 3. Two-step force transformation

To find the Lorentz transformation of force using only the simple Lorentz transformation (8), we need three reference frames: the initial laboratory frame $S$, the final laboratory frame $S'$ and $S_{\rm R}$, in which the test charge is momentarily at rest. We will boost the force first between the frames $S$ and $S_{\rm R}$, and then between the frames $S_{\rm R}$ and $S'$, and correct the obtained result for the Thomas-Wigner rotation which appears for two non-parallel boosts. The Thomas-Wigner rotation for forces is the same as for coordinates or momenta, but the boosts of forces and coordinates are different, since the force is not a four-vector. In the calculation we need the force in the frame $S$, the boost between the frames $S_{\rm R}$ and $S$ and the boost between the frames $S_{\rm R}$ and $S'$. We imagine that the force in the frame $S$, and the velocities of the frames $S$, $\mathbf{v} = c\boldsymbol{\beta}$, and $S_{\rm R}$, $\mathbf{v}_{\rm R} = c\boldsymbol{\beta}_{\rm R}$, are defined with respect to the frame $S'$. We write down the velocities in terms of the rapidity parameters and velocity directions,

$$\boldsymbol{\beta} = \hat{\boldsymbol{\beta}} \tanh\eta\,, \qquad \boldsymbol{\beta}_{\rm R} = \hat{\boldsymbol{\beta}}_{\rm R} \tanh\eta_R\,, \tag{10}$$

and the corresponding $\gamma$ parameters in terms of the rapidity parameters only,

$$\gamma \;=\; \cosh\eta\,, \qquad \gamma_{\rm R} \;=\; \cosh\eta_{\rm R}. \tag{11}$$

As an example, for coordinate boost transformation we give the coordinate transformation between $S_{\rm R}$ and $S'$ frames, $L(-\boldsymbol{\beta}_{\rm R})^{\mu}{}_{\nu}$, in the matrix form,

$$\begin{aligned} x' &= L(-\boldsymbol{\beta}_{\rm R})x_{\rm R} \\ &= \begin{pmatrix} \gamma_{\rm R} & \gamma_{\rm R}\boldsymbol{\beta}_R \\ \gamma_{\rm R}\boldsymbol{\beta}_{\rm R} & \mathbf{1} + (\gamma_{\rm R}-1)\hat{\boldsymbol{\beta}}_{\rm R}\otimes\hat{\boldsymbol{\beta}}_R \end{pmatrix} x_{\rm R}, \end{aligned} \tag{12}$$

where $\mathbf{1}$ is $3\times 3$ unit matrix and $\otimes$ denotes tensor product. The boost between $S$ and $S'$ frame is $L(-\boldsymbol{\beta})$. The parameters $\gamma_{\rm S}$ and $\boldsymbol{\beta}_S$ for boost between frames $S_{\rm R}$ and $S$ are determined by $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_{\rm R}$,

$$\begin{aligned} \gamma_{\rm S} &= \gamma_{\rm R}\gamma(1-\boldsymbol{\beta}_{\rm R}\boldsymbol{\beta})\,, \\ \gamma_{\rm S}\boldsymbol{\beta}_{\rm S} &= \gamma\boldsymbol{\beta} - \gamma\gamma_{\rm R}\boldsymbol{\beta}_{\rm R} + (\gamma_{\rm R}-1)\gamma\beta\hat{\boldsymbol{\beta}}_{\rm R}(\hat{\boldsymbol{\beta}}_{\rm R}\hat{\boldsymbol{\beta}})\,. \end{aligned} \tag{13}$$

Product of two boosts connecting three successive inertial frames in any representation of the Lorentz group can be written as a boost between the first and the last reference frame and Thomas-Wigner rotation defined by these two boosts [11]. For instance,

$$L_{\rm F}(-\boldsymbol{\beta}_{\rm R})L_F(-\boldsymbol{\beta}_{\rm S}) \;=\; L_{\rm F}(-\boldsymbol{\beta}_{\rm S})W_{\rm F}(-\boldsymbol{\beta}_{\rm R},-\boldsymbol{\beta}_{\rm S}) \tag{14}$$

shows how the two force-boosts combine into the corresponding Thomas-Wigner rotation and the boost between the first and the third reference frame. Therefore, the force in the frame $S'$ is given by

$$\mathbf{F}' \;=\; L_{\rm F}(-\boldsymbol{\beta}_{\rm R})L_{\rm F}(-\boldsymbol{\beta}_{\rm S})W_{\rm F}^{-1}(-\boldsymbol{\beta}_{\rm R},-\boldsymbol{\beta}_{\rm S})\;\mathbf{F}\,. \tag{15}$$

The force boost transformations $L_{\rm F}(-\boldsymbol{\beta}_{\rm R})$ and $L_{\rm F}(-\boldsymbol{\beta}_{\rm S})$ are defined by transformation (8) and its inverse, respectively,

$$L_{\rm F}(-\boldsymbol{\beta}_{\rm R}) = \frac{1}{\gamma_{\rm R}}\mathbf{1} + \left(1 - \frac{1}{\gamma_{\rm R}}\right)\hat{\boldsymbol{\beta}}_{\rm R}\otimes\hat{\boldsymbol{\beta}}_{\rm R}, \tag{16}$$

$$L_{\rm F}(-\boldsymbol{\beta}_{\rm S}) = \gamma_{\rm S}\mathbf{1} + (1-\gamma_{\rm S})\hat{\boldsymbol{\beta}}_{\rm S}\otimes\hat{\boldsymbol{\beta}}_{\rm S}\ . \tag{17}$$

The Thomas-Wigner rotation of forces is the same as for the coordinates,

$$\begin{aligned} W_{\rm F}(-\boldsymbol{\beta}_{\rm R}, -\boldsymbol{\beta}_{\rm S}) &= W(-\boldsymbol{\beta}_{\rm R}, -\boldsymbol{\beta}_{\rm S}) \\ &= L^{-1}(-\boldsymbol{\beta})L(-\boldsymbol{\beta}_{\rm S})L(-\boldsymbol{\beta}_{\rm R}) \\ &= \mathbf{1} + \frac{1}{\gamma_{\rm S}+1}\Big[\beta\beta_{\rm R}\gamma\gamma_{\rm R}(-\hat{\boldsymbol{\beta}}\otimes\hat{\boldsymbol{\beta}}_{\rm R} + \hat{\boldsymbol{\beta}}_{\rm R}\otimes\hat{\boldsymbol{\beta}}) \\ &\quad + (\gamma-1)(\gamma_{\rm R}-1)\Big(-\hat{\boldsymbol{\beta}}\otimes\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}_{\rm R}\otimes\hat{\boldsymbol{\beta}}_{\rm R} \\ &\quad + 2(\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}_{\rm R})\hat{\boldsymbol{\beta}}\otimes\hat{\boldsymbol{\beta}}_{\rm R}\Big)\Big]\ . \end{aligned} \tag{18}$$

The expression (18) agrees with the inverse of the expression for the Thomas-Wigner rotation derived in Ref. [12] using three-space tensor algebra techniques. The inverse Thomas-Wigner rotation (18) is obtained by exchanging $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}_{\rm R}$. Putting the expressions (16), (17) and (18) into (15) and arranging the terms, one obtains the force transformation

$$\mathbf{F}' = \gamma(1-\boldsymbol{\beta}\boldsymbol{\beta}_{\rm R})\mathbf{F} + (1-\gamma)\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{F}) + \gamma\boldsymbol{\beta}(\boldsymbol{\beta}_{\rm R}\mathbf{F}), \tag{19}$$

which is identical to the Frisch and Wilets force transformation (2), if one identifies $\boldsymbol{\beta}_{\rm R} = -\boldsymbol{\beta}'$. That is the main result of this paper.

## 4. *Discussion and comments*

The main idea of the Frisch and Wilets derivation of the force transformation is to replace the velocity of the test particle $\mathbf{v}_2$ in the standard (e.g. Møller's) expression for the force transformation,

$$\begin{aligned} \mathbf{F}' &= \frac{{\rm d}\mathbf{p}'}{{\rm d}t'} \\ &= \frac{\gamma\Big[\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{F}) - \boldsymbol{\beta}(\hat{\boldsymbol{\beta}}_2\mathbf{F})\Big] + \Big[\mathbf{F} - \boldsymbol{\beta}(\boldsymbol{\beta}_2\mathbf{F})\Big]}{\gamma(1-\boldsymbol{\beta}\boldsymbol{\beta}_2)}, \end{aligned} \tag{20}$$

with the velocity in the $S'$ frame,

$$\boldsymbol{\beta} \quad = \quad \frac{\gamma\Big[\boldsymbol{\beta} + \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\boldsymbol{\beta}'_2)\Big] + \Big[(\boldsymbol{\beta}'_2 - \boldsymbol{\beta}(\boldsymbol{\beta}\boldsymbol{\beta}'_2)\Big]}{\gamma(1 + \boldsymbol{\beta}\boldsymbol{\beta}'_2)} \,. \tag{21}$$

Rearrangement of terms leads to (2).

In the case that the postulate 2. of [1] is satisfied, as it is for the electric force in the rest frame of the point-source electric field, the force in any inertial frame can be written in terms of the force in the frame in which the force does not depend on the velocity of the test particle. For instance, for a point source $Q$ at rest in the frame $S$ acting on the point source $q$, the force in the frame $S'$ reads [1],

$$\mathbf{F}' = q\left\{\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\mathbf{E}) + \gamma\left[\hat{\boldsymbol{\beta}} \times (\mathbf{E} \times \hat{\boldsymbol{\beta}}) + \boldsymbol{\beta}'_2 \times (\mathbf{E} \times \boldsymbol{\beta})\right]\right\}, \tag{22}$$

where $\mathbf{E}$ is the electric field in the frame $S$, induced by the source charge $Q$. In (22) the postulate 1. is also assumed. In the case of several source charges, each with its rest frame $S_i$, the expression for the force reads

$$\begin{aligned} \mathbf{F}' \quad = \quad & \Big[\sum_i q_i\hat{\boldsymbol{\beta}}_i(\hat{\boldsymbol{\beta}}_i\mathbf{E}_i) + q_i\gamma_i\{\hat{\boldsymbol{\beta}}_i \times (\mathbf{E}_i \times \hat{\boldsymbol{\beta}}_i)\}\Big] \\ & + \boldsymbol{\beta}'_2 \times \Big[\sum_i q_i(\mathbf{E}_i \times \boldsymbol{\beta}_i)\Big] \,. \end{aligned} \tag{23}$$

The only quantity independent of the sources is the velocity of the test charge $\mathbf{v}'_2$. That allows to interpret the quantities in the square brackets as the electric field and magnetic field of the set of source charges moving with different velocities. It also shows that the electric and magnetic fields may be understood as a consequence of electric charges only, in agreement with Ampère's idea of magnetic fields [13]. In addition, comparing the expressions in the square brackets in two different frames $S'$, one obtains directly the boost transformation for the electric field and magnetic field given in (5) and (6), as shown in Ref. [1]. Also, that implies that electric and magnetic field form a six-component object concerning their Lorentz transformations, and therefore inevitably forms an antisymmetric tensor of rank 2.

There is also an interesting connection between the Thomas-Wigner rotations (18) and the contributions to the magnetic force $q_i(\mathbf{E}_i \times \boldsymbol{\beta}_i)$ in (23). To find it, we have to use Eqs. (15−19), replacing $\mathbf{F} \to \mathbf{F}_i = q_i\mathbf{E}_i$ and $\boldsymbol{\beta} \to \boldsymbol{\beta}_i$. One has to note that the magnetic force in (19) is contained in terms involving $\boldsymbol{\beta}_{\mathrm{R}}$, and that only the last term in (19) contains an antisymmetric term, particularly $[\gamma(\boldsymbol{\beta}\otimes\boldsymbol{\beta}_{\mathrm{R}} - \boldsymbol{\beta}_{\mathrm{R}}\otimes\boldsymbol{\beta})]\mathbf{F}$. This antisymmetric term corresponds to the antisymmetric term in the inverse of the Thomas-Wigner rotation in (18), since the other two matrices in (15) are symmetric. Therefore, the $i$-th term of the magnetic force and the corresponding contribution to the magnetic field are proportional to the antisymmetric part of the Thomas-Wigner rotation $W^{-1}(-\boldsymbol{\beta}_{\mathrm{R}}, -\boldsymbol{\beta}_i)$, which is proportional to the sine of the Thomas-Wigner-rotation angle,

$$\sin\psi_i = \frac{2\sin\phi_i(\tau_i - \cos\phi_i)}{1+\tau_i^2 - 2\tau_i\cos\phi_i}, \tag{24}$$

where $\cos\phi_i = \hat{\boldsymbol{\beta}}_i \hat{\boldsymbol{\beta}}_\mathrm{R}$ and

$$\tau_i = \left[\frac{(\gamma_\mathrm{R}+1)(\gamma_i+1)}{(\gamma_\mathrm{R}-1)(\gamma_i-1)}\right]^{1/2}. \tag{25}$$

More precisely, it is proportional to the ratio of sines

$$\frac{\sin\psi_i}{\sin\phi_i}, \tag{26}$$

because the antisymmetric part of the Thomas-Wigner rotation is proportional to the antisymmetric combination $\hat{\beta}_i \otimes \hat{\beta}_\mathrm{R} - \hat{\beta}_\mathrm{R} \otimes \hat{\beta}_i$ which is proportional to $\sin\phi_i$. As the magnetic force does contain both antisymmetric and symmetric combinations of tensor products of unit vectors $\hat{\beta}_i$ and $\hat{\beta}_\mathrm{R}$, and the symmetric one is generally proportional to unity, one obtains the result (26).

We obtained the result (24) from (18) for both velocities in the $x$-$y$ plane, and it agrees up to the sign of $\cos\phi_i$ with the corresponding result in [12]. This is the second interesting result of our paper.

## 5. *Conclusion*

We have derived the Lorentz transformation of force using only two force boost transformations from the inertial frame in which the test particle is momentarily at rest to an arbitrary inertial frame, and the corresponding Thomas-Wigner rotation. The expression for the Thomas-Wigner rotation of the coordinates and force rotation is explicitly given, and it agrees with the corresponding results in literature obtained using different calculation techniques. The force transformation is in accordance with expressions found previously in the literature, using direct boosting between initial and final frame. In our derivation, the dependence on the velocity in the final inertial frame appears naturally.

We have stressed some of the consequences of this force transformations, which are usually not connected with the Lorentz force transformation in the literature. Particularly, we showed that the magnetic field contributions are proportional to the Thomas-Wigner rotations corresponding to the pairs of boost transformations from the source rest frames, over the test-particle rest frame, to the laboratory frame.

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[13] see [2] p. 172.

## LORENTZOVA TRANSFORMACIJA SILA U DVA KORAKA

Izvodi se Lorentzova transformacija sile na prirodniji način primjenom dviju "boost" transformacija i odgovarajuće Thomas-Wignerove rotacije. Daju se neki zaključci o toj transformaciji.