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SOME NOTES ON PROJECTIVE RICCATI EQUATION METHOD

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In this paper, several aspects of projective Riccati equation method are discussed. Firstly, according to the scheme of routine projective Riccati equation method, a united scheme of projective Riccati equation method is proposed, and its mathematical foundation is studied. Secondly, some no-go theorems and positive results are obtained. The limitations of projective Riccati equation method are found. In particular, in the end, we point out that some solutions obtained by the projective Riccati equation method to some equations in a reference are not solutions at all.

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1. *Introduction*

For a given nonlinear partial differential equation (NLPDE)

$$N(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1}$$

we consider its traveling wave solutions

$$u(x,t) = u(\xi), \xi = \omega x + ct\,. \tag{2}$$

Then Eq. (1) becomes a nonlinear ordinary differential equation as

$$M(u, u', u'', \cdots) = 0\,. \tag{3}$$

A number of methods for obtaining solutions to Eq. (3) have been proposed [1 − 28]. Among those, various projective Riccati equation methods were studied extensively [18 – 28]. In Ref. [18], Conte et al. presented a general ansatz for seeking more new solitary wave solutions of some NLPDEs that can be expressed as a polynomial in two elementary functions which satisfy a projective Riccati equation [19]. Later, Yan developed Conte's method and presented the general projective Riccati eqation method [20]. Several authors have used Yan's technique to solve many NLPDNs [25 – 32]. In Ref. [25], some NLPDEs, which include single NLPDEs and coupled systems of NLPDEs, were reduced to the elliptic-like equation

$$\Phi''(\xi) + k_1\Phi(\xi) + k_3\Phi^3(\xi) = 0\,. \tag{4}$$

By Yan's technique, Emmannuel Yomba obtained sixteen kinds of exact traveling wave solutions to Eq. (4), however, we pointed out in Section 4, eleven solutions of them are not solutions at all.

In the present paper, we discuss some aspects of projective Riccati equation method. A united scheme of projective Riccati equation method is proposed, and the mathematical foundations of this method are studied. In particular, the limitations of projective Riccati equation method are found. If Eq. (3) is a nonlinear ordinary differential equation with rank homogeneous that can be reduced to an elementary integral form like Eq. (4), we can obtain all atom solutions to Eq. (3) by elementary integral method [9 – 17]. According to Ref. [16], atom solution is a solution which can be obtained by solving an integral. But the projective Riccati equation method has itself independent role for solving other form solutions such as multisolitons. In addition, some solutions obtained by projective Riccati equation method, such as eleven solutions among sixteen solutions of Eq. (4) in Ref. [25] (see Sec. 4 in this paper for details), are not solutions at all. Appearing of the added roots in solving algebraic equations by *Mathematica* maybe cause these results.

The rest of paper is arranged as follows. In Sec. 2, we describe a united scheme of the projective Riccati equation method. In Sec. 3, we obtain some theorems. In Sec. 4, we point out that some solutions given in Ref. [25] are not solutions at all. Finally we give some conclusions in Sec. 5.

2. *A united scheme of projective Riccati equation method*

According to the scheme of routine projective Riccati equation method, a united scheme is given as follows:

Step 1. We express the solution of Eq. (3) in the form

$$u(\xi) = H_1(f) + gH_2(f) = \sum_{i=0}^{m_1} b_i f^i + g\sum_{i=0}^{m_2} c_i f^i, \tag{5}$$

where $f = f(\xi)$ is a solution of the following ordinary differential equation

$$f' = fg\,, \tag{6}$$

$$g^2 = F(f) = \sum_{i=0}^{m_3} a_i f^i ,\; g' = \frac{1}{2} F'(f) f \,, \tag{7}$$

where m_1, m_2, m_3 are positive integers. By Eqs. (5), (6) and (7), we derive the following equations

$$u' = H_1{}'(f) f g + \frac{1}{2} F'(f) f H_2(f) + H_2{}'(f) f F(f) \,, \tag{8}$$

$$\begin{aligned} u'' = {} & H_1{}''(f) f^2 F(f) + H_1{}'(f) f F(f) + \frac{1}{2} H_1{}'(f) f^2 F'(f) \\ & + \frac{1}{2} F''(f) H_2(f) f^2 g + \frac{1}{2} F'(f) H_2(f) f g + \frac{3}{2} F'(f) H_2{}'(f) f^2 g \\ & + H_2{}''(f) F(f) f^2 g + H_2{}'(f) F(f) f g \,, \end{aligned} \tag{9}$$

and so on. Substituting the equations like (5), (8) and (9) into Eq. (3), and making use of Eqs. (6) and (7) yields the equation

$$G_1(f) + g G_2(f) = 0, \tag{10}$$

where $G_1(f)$ and $G_2(f)$ are two polynomials of f.

Step 2. According to the balance principle, we can obtain some relations for m_3, m_1 and m_2, from which the different possible values of m_3, m_1 and m_2 can be determined.

Step 3. Setting the coefficients of all powers of f in $G_1(f)$ and $G_2(f)$ to zeroes, we will get a system of algebraic equations, from which we will determine the values of $a_i (i = 0, 1, \cdots, m_3), b_i (i = 0, 1, \cdots, m_1)$ and $c_i (i = 0, 1, \cdots, m_2)$. However, it is possible that we will find the system hasn't a solution. This means that Eq. (3) can't be solved by the united schemes of projective Riccati equation method when m_1, m_2 and m_3 take those values.

Step 4. By Eqs. (6) and (7), we write Eq. (6) as

$$(f')^2 = f^2 F(f). \tag{11}$$

Then Eq. (11) can be reduced to the elementary integral form

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}f}{f\sqrt{F(f)}}. \tag{12}$$

Step 5. By substituting a_i $(i = 0, 1, \cdots, m_3)$ into Eq. (12), and using a complete discrimination system for m_3-th order polynomial to classify the roots of $F(f)$, we can solve Eq. (12) and obtain the exact solutions to Eq. (11). Furthermore, we can give the exact solutions to Eq. (3), respectively.

Remark 1. It is easy to see that the projective Riccati equation methods in references are special cases of the united scheme of projective Riccati equation method. For example, if we take $m_1 = n, m_2 = n-1$ and $m_3 = 2$, the united schemes of projective Riccati equations method is the projective Riccati equation method presented in Ref. [25]; if we take $m_1 = m, m_2 = m-1, m_3 = 2$, the united scheme of projective Riccati equation method is the projective Riccati equation method presented in Ref. [29], and so on.

3. *Main results*

In order to illustrate concretely and briefly our ideas, we only consider the following form of Eq. (3)

$$u'' + \alpha u' = \sum_{i=0}^{k} d_i u^i, \tag{13}$$

where α, d_i $(i = 0, 1, \cdots, k)$ are arbitrary constants, k is a positive integer. Many practice model equations, such as BBM-Burgers equations, that can be reduced to this equation. An obvious fact is that every root of $\sum\limits_{i=0}^{k} d_i u^i = 0$ is a solution of Eq. (13). We call these solutions trivial solutions. In this paper, we only consider nontrivial solutions. We will give some remarkable results of Eq. (13) in which projective Riccati equation method has some limitations to develop further. For Eq. (13), by step 1 and step 2, we can obtain some relations for m_1, m_2 and m_3. For example, if $k = 2$, then $m_2 \leq m_1/2$ and $m_3 = m_1$; if $k = 3$, then $m_2 = 0$ and $1 \leq m_1 \leq m_3/2$. Thus, we can determine the different possible values of m_1, m_2 and m_3. Among those values of m_1, m_2 and m_3, when we take some of them, Eq. (13) can be solved by using the united schemes of projective Riccati equation method. But Eq. (13) can't be solved when we take others.

Theorem 1: Eq. (13) has nontrivial solutions by the united schemes of projective Riccati equations method at least in the following cases:

Case 1: $k = 2$, and (1) $m_3 = m_1 = 1, m_2 = 0$; or (2) $m_3 = m_1 = 4, m_2 = 0$;

Case 2: $k = 3$, and (1) $m_3 = 2, m_1 = 1, m_2 = 0$; or (2) $m_3 = 4, m_1 = 2, m_2 = 0$.

Proof: For brevity, we only prove (1) of Case 2 . When $k = 3$, Eq. (13) becomes

$$u'' + \alpha u' = d_3 u^3 + d_2 u^2 + d_1 u + d_0. \tag{14}$$

Substituting Eqs. (5), (8) and (9) into (14) and use of (6) and (7) yields Eq. (10), where

$$G_1(f) = {H_1}''(f) f^2 F(f) + {H_1}'(f) f F(f) + \frac{1}{2} {H_1}'(f) f^2 F'(f)$$
$$+ \frac{1}{2} \alpha F'(f) f H_2(f) + \alpha {H_2}'(f) f F(f) - d_3 [H_1(f)]^3 - d_2 [H_1(f)]^2$$

$$-3d_3F(f)H_1(f)[H_2(f)]^2 - d_2F(f)[H_2(f)]^2 - d_1H_1(f) - d_0\,, \tag{15}$$

$$G_2(f) = \frac{1}{2}F''(f)H_2(f)f^2 + \frac{1}{2}F'(f)H_2(f)f + \frac{1}{2}F'(f){H_2}'(f)f^2$$
$$+{H_2}''(f)F(f)f^2 + {H_2}'(f)F(f)f + {H_2}'F'(f)f^2 + \alpha {H_1}'(f)f$$
$$-3d_3[H_1(f)]^2H_2(f) - d_3F(f)[H_2(f)]^3 - 2d_2H_1(f)H_2(f) - d_1H_2(f)\,. \tag{16}$$

By step 2, we know that m_3 can be taken 2, m_1 can be taken 1 and m_2 can be taken 0, that is, $m_3 = 2, m_1 = 1, m_2 = 0$. So we have

$$H_1(f) = \sum_{i=0}^{1} b_i f^i, H_2(f) = c_0, F(f) = \sum_{i=0}^{2} a_i f^i. \tag{17}$$

Substituting (17) into (15) and (16) and by step 3, setting the coefficients of all powers of f to zero, we get the following system of algebraic equations:

$$2a_2c_0 - 3d_3{b_1}^2c_0 - d_3a_2{c_0}^3 = 0, \tag{18}$$

$$\frac{1}{2}c_0a_1 + \alpha b_1 - 6d_3b_1b_0c_0 - d_3a_1{c_0}^3 - 2d_2b_1c_0 = 0, \tag{19}$$

$$3d_3{b_0}^2c_0 + d_3a_0{c_0}^3 + 2d_2b_0c_0 + d_1c_0 = 0, \tag{20}$$

$$2b_1a_2 - d_3{b_1}^3 - 3{c_0}^2d_3a_2b_1 = 0, \tag{21}$$

$$\frac{3}{2}b_1a_1 + \alpha a_2c_0 - 3d_3{b_1}^2b_0 - 3d_3{c_0}^2a_2b_0 - 3d_3{c_0}^2a_1b_1 - d_2{b_1}^2 - d_2a_2{c_0}^2 = 0, \tag{22}$$

$$b_1a_0+\frac{1}{2}\alpha c_0a_1-3d_3b_1{b_0}^2-3d_3{c_0}^2a_0b_1-3d_3{c_0}^2a_1b_0-2d_2b_1b_0-d_2a_1{c_0}^2-d_1b_1=0\,, \tag{23}$$

$$d_3{b_0}^3 + 3d_3{c_0}^2a_0b_0 + d_2{b_0}^2 + d_2a_0{c_0}^2 + d_1b_0 + d_0 = 0. \tag{24}$$

From Eqs. (18) – (24) we can obtain the following solutions:

$${c_0}^2 = \frac{1}{2d_3}, b_0 = \frac{\alpha - 2d_2c_0}{6d_3c_0}, a_0 = \frac{2{d_2}^2 - \alpha^2d_3 - 6d_1d_3}{3d_3}, a_2 = 2d_3{b_1}^2, \tag{25}$$

where b_1 is arbitrary nonzero constant, a_1 is arbitrary constant, and α, d_3, d_2, d_1, d_0 satisfy the expression: $2\alpha^3d_3 - 3\alpha{d_2}^2 + 9\alpha d_1d_3 = c_0(2{d_2}^3 - 9d_1d_2d_3 + 27d_0{d_3}^2)$.

Thus, Eq. (13) has nontrivial solutions when $k = 3$ and $m_3 = 2, m_1 = 1, m_2 = 0$. The proof is completed.

Now, we give some solutions of Eq. (14) when $m_3 = 2, m_1 = 1, m_2 = 0$. The solutions of Eq. (14) can be given the following form according to Eqs. (5) and (7)

$$u(\xi) = b_1 f + b_0 \pm c_0 \sqrt{a_2 f^2 + a_1 f + a_0}, \tag{26}$$

where f can be taken as arbitrary solution of Eq. (12) of the following cases:

$$\pm \frac{a_1}{2\sqrt{a_2}}(\xi - \xi_0) = \ln \left| \frac{f - a_1/(2a_2)}{f} \right|, \ (a_2 > 0), \tag{27}$$

$$\pm\sqrt{a_2}(\xi - \xi_0) = \frac{1}{\sqrt{\beta\gamma}} \ln \frac{[\sqrt{(-\gamma)(f-\beta)} - \sqrt{(-\beta)(f-\gamma)}]^2}{|f|}, \ (a_2 > 0), \tag{28}$$

$$\pm\sqrt{a_2}(\xi - \xi_0) = \frac{1}{\sqrt{\beta\gamma}} \ln \frac{[\sqrt{\gamma(f-\beta)} - \sqrt{(\beta)(f-\gamma)}]^2}{|f|}, \ (a_2 > 0), \tag{29}$$

$$\pm\sqrt{a_2}(\xi - \xi_0) = \frac{1}{\sqrt{-\beta\gamma}} \arcsin \frac{(-\gamma)(f-\beta) + (-\beta)(f-\gamma)}{|f||\beta-\gamma|}, \ (a_2 > 0), \tag{30}$$

$$\pm\sqrt{(-a_2)}(\xi - \xi_0) = \frac{1}{\sqrt{-\beta\gamma}} \ln \frac{[\sqrt{(-\gamma)(f+\beta)} - \sqrt{(\beta)(f-\gamma)}]^2}{|f|}, \ (a_2 < 0), \tag{31}$$

$$\pm\sqrt{(-a_2)}(\xi - \xi_0) = \frac{1}{\sqrt{-\beta\gamma}} \ln \frac{[\sqrt{\gamma(-f+\beta)} - \sqrt{(-\beta)(f-\gamma)}]^2}{|f|}, \ (a_2 < 0), \tag{32}$$

$$\pm\sqrt{(-a_2)}(\xi - \xi_0) = \frac{1}{\sqrt{\beta\gamma}} \arcsin \frac{(-\gamma)(f+\beta) + \beta(f-\gamma)}{|f||\beta-\gamma|}, \ (a_2 < 0), \tag{33}$$

$$\pm a_0(\xi - \xi_0) = \ln \left| \frac{-f/(2a_1\sqrt{a_0}) + \sqrt{a_0} - \sqrt{a_2 f^2 + a_1 f + a_0}}{f} \right|, \ (a_0 > 0), \tag{34}$$

where $\beta = \dfrac{-a_1 + \sqrt{{a_1}^2 - 4a_2 a_0}}{2a_2}$, $\gamma = \dfrac{-a_1 - \sqrt{{a_1}^2 - 4a_2 a_0}}{2a_2}$.

For example, substituting Eq. (27) into Eq. (26), a solution of Eq. (14) can be obtained as follows

$$u_1(\xi) = \pm c_0 \sqrt{a_2 \left(\frac{\frac{a_1}{2a_2}}{1 \mp e^{\pm \frac{a_1}{2\sqrt{a_2}}(\xi-\xi_0)}} \right)^2 + a_1 \left(\frac{\frac{a_1}{2a_2}}{1 \mp e^{\pm \frac{a_1}{2\sqrt{a_2}}(\xi-\xi_0)}} \right) + a_0} + \frac{\frac{a_1 b_1}{2a_2}}{1 \mp e^{\pm \frac{a_1}{2\sqrt{a_2}}(\xi-\xi_0)}} + b_0. \tag{35}$$

Theorem 2. Eq. (13) has not nontrivial solutions by the united schemes of projective Riccati equation method at least in the following cases :

Case 1: $k = 2$, and (1) $m_3 = m_1 = 3, m_2 = 1$; or (2) $m_3 = m_1 = 4, m_2 = 1$;

Case 2: $k = 3$, and (1) $m_3 = 3, m_1 = 1, m_2 = 0$; or (2) $m_3 = 4, m_1 = 1, m_2 = 0$.

Proof: For brevity, we only prove (1) of Case 2.
When $k = 3$, Eq. (13) becomes Eq. (14). Correspondingly, we have

$$H_1(f) = \sum_{i=0}^{1} b_i f^i, \quad H_2(f) = c_0, \quad F(f) = \sum_{i=0}^{3} a_i f^i. \tag{36}$$

By step 3, we get the following system of algebraic equations:

$$\frac{9}{2} a_3 c_0 - d_3 a_3 {c_0}^3 = 0\,, \tag{37}$$

$$2a_2 c_0 - 3d_3 {b_1}^2 c_0 - d_3 a_2 {c_0}^3 = 0\,, \tag{38}$$

$$\frac{1}{2} a_1 c_0 + \alpha b_1 - 6d_3 b_1 b_0 c_0 - d_3 {c_0}^3 a_1 - 2d_2 b_1 c_0 = 0\,, \tag{39}$$

$$2d_2 b_0 c_0 + d_1 c_0 + 3d_3 {b_0}^2 c_0 + d_3 {c_0}^3 a_0 = 0\,, \tag{40}$$

$$\frac{5}{2} b_1 a_3 - 3d_3 a_3 b_1 {c_0}^2 = 0\,, \tag{41}$$

$$2b_1 a_2 + \frac{3}{2} \alpha c_0 a_3 - d_3 {b_1}^3 - 3d_3 a_3 b_0 {c_0}^2 - 3d_3 a_2 b_1 {c_0}^2 - d_2 a_3 {c_0}^2 = 0\,, \tag{42}$$

$$\frac{3}{2} b_1 a_1 + \alpha a_2 c_0 - 3d_3 b_0 {b_1}^2 - 3d_3 a_2 b_0 {c_0}^2 - 3d_3 a_1 b_1 {c_0}^2 - d_2 {b_1}^2 - d_2 a_2 {c_0}^2 = 0\,, \tag{43}$$

$$b_1 a_0 + \frac{1}{2} \alpha a_1 c_0 - 3d_3 b_1 {b_0}^2 - 3d_3 a_1 b_0 {c_0}^2 - 3d_3 a_0 b_1 {c_0}^2 - 2d_2 b_1 b_0 - d_2 a_1 {c_0}^2 - d_1 b_1 = 0\,, \tag{44}$$

$$d_3 {b_0}^3 + 3d_3 a_0 b_0 {c_0}^2 + d_2 {b_0}^2 + d_2 a_0 {c_0}^2 + d_1 b_0 + d_0 = 0\,. \tag{45}$$

According to Eq. (37) and Eq. (41), we find ${c_0}^2 = 9/(2d_3)$ and ${c_0}^2 = 5/(6d_3)$, what is conflictive because a_3, c_0 and d_3 are all nonzero positive integers. So this system of algebraic equations has not solutions, thus Eq. (13) has not nontrivial solutions by the united scheme of projective Riccati equation method when $k = 3$ and $m_3 = 3, m_1 = 1, m_2 = 0$. Other cases can be proved similarly. The proof is completed.

Theorem 3. If $k \geq 4$, then Eq. (13) can't be solved by the united schemes of projective Riccati equation method.

Proof: If $k \geq 4$, it is easy to see that whatever m_3, m_1 and m_2 be chosen any non-negative integer, the highest derivative term and the highest nonlinear terms of $G_1(f)$ and $G_2(f)$ can't be balanced.

Remark 2. From the above results, the limitations of the projective Riccati equation method are seen.

Remark 3. The united scheme of the projective Riccati equation method proposed above can also be developed when we take $H_1(f),\ H_2(f)$ and $F(f)$ rational function forms in Eqs. (5) and (7).

4. *Discussion of solutions to Eq. (4)*

In this section, we list all atom solutions to Eq. (4). In particular, we point out that many solutions to Eq. (4) in Ref. [25] are not solutions at all.

Integrating Eq. (4) twice, we obtain

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}\Phi}{\sqrt{-k_1\Phi^2 - \frac{k_3}{2}\Phi^4 + D}}, \tag{46}$$

where D is an integral constant. By elementary integral method [9−17], we can obtain all atom solutions of Eq. (4) as follows:

$$\Phi_1(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\tanh\left(\sqrt{\frac{k_1}{2}}(\xi - \xi_0)\right), \tag{47}$$

$$\Phi_2(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\coth\left(\sqrt{\frac{k_1}{2}}(\xi - \xi_0)\right), \tag{48}$$

$$\Phi_3(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\tan\left(\sqrt{-\frac{k_1}{2}}(\xi - \xi_0)\right), \tag{49}$$

$$\Phi_4(\xi) = \pm\frac{2}{(-2k_3)^{1/2}(\xi - \xi_0)}, \tag{50}$$

$$\Phi_5(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\left[\tanh^2(\sqrt{-\frac{k_1}{2}}(\xi-\xi_0)) - 2\right]^{\frac{1}{2}}, \tag{51}$$

$$\Phi_6(\xi) = \pm\sqrt{\frac{k_1}{k_3}}\left[\coth^2(\sqrt{-\frac{k_1}{2}}(\xi-\xi_0)) - 2\right]^{\frac{1}{2}}, \tag{52}$$

$$\Phi_7(\xi) = \pm\sqrt{-\frac{k_1}{k_3}}\left[\tan^2(\sqrt{\frac{k_1}{2}}(\xi-\xi_0)) + 2\right]^{\frac{1}{2}}, \tag{53}$$

$$\Phi_8(\xi) = \pm(-2k_3)^{-1/6}\left[\alpha + (\beta-\alpha)\ \mathrm{sn}^2(\frac{\sqrt{\gamma-\alpha}}{2}(-2k_3)^{1/3}(\xi-\xi_0), m)\right]^{1/2},$$
$$(\alpha<\beta<\gamma, m^2 = \frac{\beta-\alpha}{\gamma-\alpha}), \tag{54}$$

$$\Phi_9(\xi) = \pm(-2k_3)^{-1/6}\left[\frac{-\beta\ \mathrm{sn}^2(\sqrt{\gamma-\alpha}((-2k_3)^{1/3}\xi-\xi_0)/2, m) + \gamma}{\mathrm{cn}^2(\sqrt{\gamma-\alpha}((-2k_3)^{1/3}\xi-\xi_0)/2, m)}\right]^{\frac{1}{2}},$$
$$(\alpha<\beta<\gamma,\ m^2 = \frac{\beta-\alpha}{\gamma-\alpha}), \tag{55}$$

$$\Phi_{10}(\xi) = \pm\left(\frac{2D}{-k_3}\right)^{-1/4}\left[\frac{2}{1+\mathrm{cn}((-8Dk_3)^{\frac{1}{4}}(\xi-\xi_0), m)} - 1\right]^{\frac{1}{2}}. \tag{56}$$

In the following, we point out that many solutions in Ref. [25] are not solutions at all.

Using the projective Riccati equation method in Ref. [25] (in fact, the method is a special case of our in this paper proposed united scheme projective Riccati equation method with $m_1 = n, m_2 = n-1, m_3 = 2$), Emmanuel Yomba obtained sixteen kinds of solutions to Eq. (4): $\Phi_1, \Phi_2, \cdots, \Phi_{16}$ (see Ref. [25] for details). Unfortunately, we find many faults in these solutions.

The solutions $\Phi_3, \Phi_4, \Phi_7, \Phi_8, \cdots, \Phi_{15}$ in Ref. [25] do not satisfy Eq. (4) at all, which can be easily proved by simple calculations. For example, in Ref. [25],

$$\Phi_3 = \pm\sqrt{-\frac{2}{k_3}}\mathrm{sech}(\sqrt{-k_1}\,\xi), \tag{57}$$

so, we have

$$\Phi_3'' = \pm(-k_1)\sqrt{-\frac{2}{k_3}}\ \frac{-\cosh^2(\sqrt{-k_1}\xi) + 2\sinh^2(\sqrt{-k_1}\xi)}{\cosh^3(\sqrt{-k_1}\xi)}, \tag{58}$$

substituting Eqs. (57) and (58) into Eq. (4) yields

$$\pm 2(k_1 - 1)\frac{\sqrt{-2/k_3}}{\cosh^3(\sqrt{-k_1}\xi)} = 0, \tag{59}$$

From Eq. (59), we must have $k_1 = 1$. Hence, the solution Φ_3 in Ref. [25] doesn't satisfy Eq. (4).

Using similar method, we can prove that $\Phi_4, \Phi_7, \Phi_8, \cdots, \Phi_{15}$ in Ref. [25] do not satisfy Eq. (4), and hence they are not solutions of Eq. (4) at all.

Thus, only five kinds of solutions obtained in Ref. [25] are real solutions of Eq. (4). They are $\Phi_1 = \pm\sqrt{-k_1/k_3}\tanh(\sqrt{k_1/2}\,\xi)$, $\Phi_2 = \pm\sqrt{-k_1/k_3}\coth(\sqrt{k_1/2}\,\xi)$, $\Phi_5 = \pm\sqrt{k_1/k_3}\tan(\sqrt{-k_1/2}\,\xi)$, $\Phi_6 = \pm\sqrt{k_1/k_3}\cot(\sqrt{-k_1/2}\,\xi)$, $\Phi_{16} = \pm\sqrt{-2/k_3}\,1/\xi$. Obviously, all of them belong to the solutions listed by us above.

Remark 4. In Eq. (49), if we let $\xi_0 = 0$, then $\Phi_3(\xi) = \pm\sqrt{k_1/k_3}\tan(\sqrt{-k_1/2}\,\xi)$, which is just Φ_5 in Ref. [25]. If we let $\xi_0 = \pi/\sqrt{-2k_1}$, then $\Phi_3(\xi) = \pm\sqrt{k_1/k_3}\cot(\sqrt{-k_1/2}\,\xi)$, which is just Φ_6 in Ref. [25]. That is, Φ_5 and Φ_6 in Ref. [25] are special cases of Eq. (49).

Remark 5. As noted by the referee, a solution of Eq. (4) that is not an atom solution is given by the projective method

$$\Phi(\xi) = -\frac{\sqrt{k_1}A\sinh\left(\sqrt{2k_1}\,\xi\right) - \sqrt{k_1}B\cosh\left(\sqrt{2k_1}\,\xi\right) + 8a_1\sqrt{-k_2}\,k_1\rho^2}{\sqrt{-k_2}\left(B\sinh(\sqrt{2k_1}\,\xi) - A\cosh(\sqrt{2k_1}\xi) - 2\beta\rho^4\right)}, \tag{60}$$

where $a_1, \beta, \rho \neq 0$ are constants and $A = 16{a_1}^2k_1k_2 + (\beta^2+1)\beta^4$, $B = 16{a_1}^2k_1k_2 + (\beta^2-1)\beta^4$. The solution (60) has an equivalent form

$$\Phi(\xi) = A\frac{e^{\sqrt{2k_1}\xi} + Be^{-\sqrt{2k_1}\xi} + C}{e^{\sqrt{2k_1}\xi} - Be^{-\sqrt{2k_1}\xi} + D}, \tag{61}$$

where A, B, C, D are constants. This solution can not been obtained by the direct integral method. It can be considered as a doubled solitary solution with two opposite traveling directions.

5. *Conclusions*

In summary, we give a united scheme of projective Riccati equation method, and discuss the mathematical foundations of this method. We find the limitations of the projective Riccati equation method, e.g., this method does not work for some rank inhomogeneous nonlinear ordinary differential equations in which the order of polynomial is bigger than three. In addition, we point out that some solutions of Eq. (4) given in Ref. (25) aren't solutions at all.

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NEKE NAPOMENE O PROJEKTIVNOJ METODI ZA RICCATIJEVU JEDNADŽBU

U ovom radu raspravljamo više primjena projektivne metode za rješavanje Riccatijeve jednadžbe. Prvo, prema uobičajenom pristupu te metode, predlažemo zajednički pristup za projektivno rješavanje Riccatijeve jednadžbe i proučavamo njegove matematičke osnove. Zatim se daju neki teoremi kada nema rezultata kao i neka rješenja. Nađena su i ograničenja projektivne metode za rješavanje Riccatijeve jednadžbe. Na kraju, pokazujemo kako neka rješenja nekih jednadžbi dobivena tom metodom nisu dobra rješenja.