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FINDING THE BELL-TYPE SOLITON SOLUTIONS TO NONLINEAR EVOLUTION EQUATIONS BY A UNIFIED METHOD

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A simple and unified method is presented to find the bell-type soliton solutions for nonlinear evolution equations by introducing a transformation containing two unknown functions and selecting appropriately trial functions. We demonstrate its effectiveness by applications to certain physically significant equations as particular examples. The technique used herein can also be used to explore the bell-type soliton solutions of other nonlinear evolution equations.

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1. *Introduction*

Many phenomena in physics and other fields are often described by nonlinear evolution equations (NEEs). When people want to understand the physical mechanism of natural phenomena described by NEEs, explicit exact solutions of NEEs have to be explored, and thus investigation of the explicit exact solutions of NEEs plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by bell and kink shaped soliton solutions, and the vibration of masses in a lattice with exponential interaction force is often modelled by the travelling wave solutions of the Toda lattice. In recent years, a vast variety of powerful and efficient methods have been established and developed to construct the explicit exact solutions of NEEs. Among them are the homogeneous balance method [1, 2], the hyperbolic tangent function expansion method [3-6], the trial function method [7 − 11], the sine-cosine method [12], the Jacobi elliptic function expansion method

[13, 14], the superposition method [15], the auxiliary equation method [16 – 19], and so on. Nevertheless, there exists no unified method for solving NEEs. As a result, it is still a very challenging task to seek new and more powerful methods to solve NEEs.

In the present paper, by introducing a transformation containing two unknown functions and selecting appropriately trial functions, we successfully derive the bell-type soliton solutions of some physically important NEEs in a simple and unified manner.

2. *Essentials of the approach*

The fundamental idea of our approach is as follows. Consider a given NEE, say, in two independent variables x and t, such as

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0\,. \tag{1}$$

Generally speaking, the left-hand side of Eq. (1) is a polynomial in terms of unknown function $u(x, t)$ and its various partial derivatives.

The primary aim of the present paper is to construct the bell-type soliton solution of Eq. (1) in a systematic and unified way. To this end, we introduce a transformation in the form

$$u = u_0 + \frac{\partial u}{\partial x}, \quad w = \frac{1}{w}\frac{\partial w}{\partial x}, \quad w = w(y), \quad y = y(x, t)\,, \tag{2}$$

where $w(y)$ and $y(x, t)$ are two functions to be determined later, and u_0 is an undetermined constant.

To begin with, let us determine the trial function $y = y(x, t)$. It is well known that the travelling wave solution of NEEs should contain the factor $k(x - ct)$. Therefore, we directly choose the trial function $y = y(x, t)$ of the following form

$$y = \mathrm{e}^{k(x-ct)}\,, \tag{3}$$

where k and c are the wave number and wave speed, respectively.

For the other trial function $w(y)$, after our careful and repeated considerations, we select it in the following form

$$w(y) = (a + y^2)^b\,, \tag{4}$$

where a and b are constants to be determined later.

Based on Eqs. (2) – (4), it is easily deduced that

$$v = \frac{1}{w}\frac{\partial w}{\partial x} = \frac{2bky^2}{a + y^2}\,, \tag{5}$$

and

$$u = u_0 + \frac{\partial v}{\partial x} = u_0 + \frac{4abk^2y^2}{(a+y^2)^2}\,, \tag{6}$$

Now, let us simply describe our procedure. Firstly, substituting Eq. (6) and Eq. (3) into Eq. (1) engenders a set of algebraic equations because of the coefficients of all powers of y must vanish. Secondly, by solving the obtained system of algebraic equations, we get the unknown constant b. Lastly, putting b into Eq. (6) and utilizing Eq. (3) together with the setting $a = \mathrm{e}^{-2kx_0}$, the bell-type soliton solutions of Eq. (1) are obtained. In the following, we would like to employ the approach stated above to find the bell-type soliton solutions of certain physically significant NEEs.

3. *Applications*

3.1. *Benjamin Ono equation*

The celebrated Benjamin Ono equation with the dispersive term reads

$$\frac{\partial^2 u}{\partial t^2} + 2p\left(\frac{\partial u}{\partial x}\right)^2 + 2pu\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^4 u}{\partial x^4} = 0\,, \tag{7}$$

where p and q are arbitrary real constants.

Plugging Eq. (6) and Eq. (3) into Eq. (7), and collecting the coefficients of powers of y with the aid of Mathematica, then setting each of the obtained coefficients to zero, yields a set of algebraic equations with respect to the unknown constants a and b as follows

$$16a^5bc^2k^4 + 32a^5bk^4pu_0 + 64a^5bk^6q = 0\,, \tag{8}$$

$$-32a^4bc^2k^4 - 64a^4bk^4pu_0 + 256a^4b^2k^6p - 1664a^4bk^6q = 0\,, \tag{9}$$

$$-96a^3bc^2k^4 - 192a^3bk^4pu_0 - 768a^3b^2k^6p + 4224a^3bk^6q = 0,, \tag{10}$$

$$-32a^2bc^2k^4 - 64a^2bk^4pu_0 + 256a^2b^2k^6p - 1664a^2bk^6q = 0,, \tag{11}$$

$$16abc^2k^4 + 32abk^4pu_0 + 64abk^6q = 0\,. \tag{12}$$

Solving the above set of algebraic equations by Mathematica, we obtain

$$u_0 = -\frac{c^2+4k^2q}{2p}, \quad b = \frac{6q}{p}, \quad a = \text{an arbitrary constant.} \tag{13}$$

Putting Eq. (13) into Eq. (6) and considering Eq. (3) simultaneously, we arrive at

$$u = -\frac{c^2 + 4k^2 q}{2p} + \frac{24ak^2 q \exp(2k(x-ct))}{p\big[a + \exp(2k(x-ct))\big]^2}\,. \tag{14}$$

Setting $a = \mathrm{e}^{-2kx_0}$ in Eq. (14) and applying the following identity

$$\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+1} = \frac{1}{2}\,\mathrm{sech}\;x\,, \tag{15}$$

we obtain the bell-type soliton solution of the Benjamin Ono Eq. (7) as follows

$$u = -\frac{c^2 + 4k^2 q}{2p} + \frac{6k^2 q}{p}\mathrm{sech}^2\,k(x - ct + x_0)\,, \tag{16}$$

where x_0 has a definite physical significance, namely, it represents the center of the soliton.

3.2. Fifth-order nonlinear equation

Consider the following fifth-order nonlinear equation

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} - \frac{\partial^5 u}{\partial x^5} = 0\,. \tag{17}$$

Inserting Eq. (6) and Eq. (3) into Eq. (17), and collecting the coefficients of powers of y with the help of Mathematica, then setting each of the obtained coefficients to zero, results in a set of algebraic equations with regard to the unknown constants a and b as follows

$$-8a^6bck^3 + 8a^6bk^3u_0^2 - 128a^6bk^3 = 0\,, \tag{18}$$

$$-24a^5bck^3 + 24a^5bk^3u_0^2 + 64a^5b^2k^5u_0 + 7296a^5bk^7 = 0\,, \tag{19}$$

$$-16a^4bck^3 + 16a^4bk^3u_0^2 + 64a^4b^2k^5u_0 - 38656a^4bk^7 + 128a^4b^3k^7 = 0\,, \tag{20}$$

$$16a^3bck^3 - 16a^3bk^3u_0^2 - 64a^3b^2k^5u_0 + 38656a^3bk^7 - 128a^3b^3k^7 = 0\,, \tag{21}$$

$$24a^2bck^3 - 24a^2bk^3u_0^2 - 64a^2b^2k^5u_0 - 7296a^2bk^7 = 0\,, \tag{22}$$

$$8abck^3 - 8abk^3u_0^2 + 128abk^7 = 0\,. \tag{23}$$

Solving the above system of algebraic equations by Mathematica, we obtain

$$u_0 = \mp\sqrt{10}\,k^2, \quad b = \pm\sqrt{10}, \quad a = \text{an arbitrary constant.} \tag{24}$$

Plugging Eq. (24) into Eq. (6) and considering Eq. (3) simultaneously yields

$$u = \mp\sqrt{10}\,k^2 \pm \frac{24\sqrt{10}\,ak^2 \mathrm{e}^{2k(x-ct)}}{\left[a + \mathrm{e}^{2k(x-ct)}\right]^2} \,. \tag{25}$$

Similarly, setting $a = \mathrm{e}^{-2kx_0}$ in Eq. (25) and utilizing the foregoing identity (15), we gain the bell-type soliton solution to the fifth-order nonlinear equation (17) as follows

$$u = \mp 2\sqrt{10}\,k^2 \pm 6\sqrt{10}\,k^2 \mathrm{sech}^2\, k(x - ct + x_0)\,, \tag{26}$$

3.3. UNSO-KdV equation

The UNSO-KdV equation reads

$$\frac{\partial u}{\partial t} + \frac{4}{147}u^3\frac{\partial u}{\partial x} + \frac{5}{14}\left(\frac{\partial u}{\partial x}\right)^3 + \frac{9}{7}u\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{2}{7}u^2\frac{\partial^3 u}{\partial x^3} + 6\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3}$$
$$+\frac{7}{2}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + u\frac{\partial^5 u}{\partial x^5} + \frac{\partial^7 u}{\partial x^7} = 0\,, \tag{27}$$

which was first proposed in Ref. [20].

Substituting Eq. (6) and Eq. (3) into Eq. (27), and collecting the coefficients of powers of y with the aid of Mathematica, then setting each of the obtained coefficients to zero, brings about a set of algebraic equations with respect to the unknown constants a and b as follows

$$32bnk^3u_0^3 + 1344abk^5u_0^2 + 18816abk^7u_0 + 75264abk^9 - 1176abck^3 = 0\,, \tag{28}$$

$$\begin{gathered}160a^2bk^3u_0^3 - 9408a^2bk^5u_0^2 - 1034880a^2bk^7u_0 - 18590208a^2bk^9\\ +384a^2b^2k^5u_0^2 + 34944a^2b^2k^7u_0 + 790272a^2b^2k^9 + 5880a^2bck^3 = 0\,,\end{gathered} \tag{29}$$

$$\begin{gathered}288a^3bk^3u_0^3 - 36288a^3bk^5u_0^2 + 35562240a^3bk^7u_0 + 323108352a^3bk\\ 1152a^3b^2k^5u_0^2 - 169344a^3b^2k^7u_0 - 18176256a^3b^2k^9 + 1536a^3b^3k^7u_0\\ -801024a^4b^3k^9 + 2048a^4b^4k^9 - 5880a^4bck^3 = 0\,,\end{gathered} \tag{30}$$

$$\begin{gathered}160a^4bk^3u_0^3 - 25536a^4bk^5u_0^2 + 4609920a^4bk^7u_0 - 1175548416a^4bk^9\\ +768a^4b^2k^5u_0^2 - 204288a^4b^2k^7u_0 + 72253440a^4b^2k^9 + 1536a^4b^3k^7u_0 = 0\,.\end{gathered} \tag{31}$$

It follows from Eqs. (28) – (31) that

$$u_0 = -7k^2, \quad b = 21, \quad a = \text{an arbitrary constant.} \tag{32}$$

Plugging Eq. (32) into Eq. (6) and considering Eq. (3) simultaneously, we obtain

$$u = -7k^2 + \frac{84ak^2 e^{2k(x-ct)}}{\left[a + e^{2k(x-ct)}\right]^2} \,. \tag{33}$$

Similarly, setting $a = e^{-2kx_0}$ in Eq. (33), and making use of the previous identity (15), we acquire the bell-type soliton solution for the UNSO-KdV equation (27) as follows

$$u = -7k^2 + 21k^2 \mathrm{sech}^2\, k(x - ct + x_0) \,. \tag{34}$$

4. *Conclusions*

In conclusion, by introducing a transformation which contains two unknown functions and selecting appropriately trial functions, we develop a simple and unified method to seek the bell-type soliton solutions of NEEs, and some illustrative equations are investigated by this means. Its advantage is that one can easily find the bell-type soliton solutions, if they exist, for NEEs under consideration in a systematic and unified way, and thus it is readily and conveniently applied to explore the bell-type soliton solutions of other NEEs.

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NALAŽENJE ZVONASTIH SOLITONSKIH RJEŠENJA NELINEARNIH EVOLUCIJSKIH JEDNADŽBI JEDINSTVENOM METODOM

Predstavljamo jednostavan i jedinstven način nalaženja zvonastih solitonskih rješenja nelinearnih evolucijskih jednadžbi uvođenjem pretvorbe koja sadrži dvije nepoznate funkcije i odabirom pogodnih probnih funkcija. Pokazujemo učinkovitost metode primjenom na nekoliko važnih jednadžbi fizike kao posebnim primjerima. Ova se metoda može primijeniti također za nalaženje zvonastih solitonskih rješenja drugih nelinearnih evolucijskih jednadžbi.