Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

THE JONES-MUELLER TRANSFORMATION

CARLOS HUNTE

University of The West Indies, Cave Hill Campus, The Department of Computer Science, Mathematics and Physics, P.O. Box 64, Bridgetown, Barbados, West Indies

Received 21 March 2007; Accepted 14 May 2008
Online 25 September 2008

A simple transformation allows the construction of the real-valued Mueller matrix from the complex-valued Jones matrix. The transformation is tested on the Jones matrices of optical devices to produce the corresponding Mueller matrices. The Stokes vector is also deduced from the corresponding Jones vector.

PACS numbers: 42.00.00, 42.25.-p UDC 535.51
Keywords: polarization of light, Jones matrix, Mueller matrix, unitary transformation

1. *Introduction*

A Mueller matrix is suited to all types of optical systems, while a Jones matrix completely characterizes the polarization properties of non-depolarizing optical systems. Hence, for a non-depolarizing optical system, Jones matrices and Mueller matrices are equivalent.

When the two matrices are equivalent, one matrix is usually preferred over the other in some situations. A Jones matrix has fewer elements and the physical meanings of the matrix elements are clearer. A Mueller matrix uses only real numbers, and the intensity transformation property of a sample is expressed in its M_{00} element and, therefore, can provide an image of a sample without the influence of its polarization property. Thus, a Mueller matrix separates structural information from polarization information of a sample.

To determine the Mueller matrix of a sample or device, sixteen combinations of source and analyzing polarizations are usually employed. This is relatively time consuming and, hence, is of limited application with the use of unstable samples. To measure the Mueller matrix of such samples, it would be advantageous to measure the Jones matrix of the sample and then determine the Mueller matrix by a suitable transformation. This technique can be employed in the optical coherence tomography (OCT) [1] of soft biological tissues where the measured Jones matrix can be transformed into an equivalent Mueller matrix. OCT techniques have been used to image skin tissue, gastrointestinal and respiratory tracts and cervical dysplasia and carcinoma.

The purpose of this article is to provide a mechanism that allows for the transformation of a Jones matrix of an optical device into an equivalent Mueller matrix for that device.

It is important to set up a coordinate system in order to define the Jones vectors and matrices. We use the standard convention in optics, shown in Fig. 1, where the observer looks towards the source and measures positive as anticlockwise. Figure 2 shows the orientation of left-circularly polarized light and right-circularly polarized light in the coordinate system employed. Unfortunately, this is not the convention adopted in other areas of physics and, hence, confusion arises pertaining to parameters in optical physics. For example, Hecht [2] and Pedrotti [3] give the Jones matrix for a quarter wave plate (QWP) with fast axis (FA) horizontal as $\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$, whereas Fowles [4] and Meyer-Arendt [5] give the Jones matrix for the

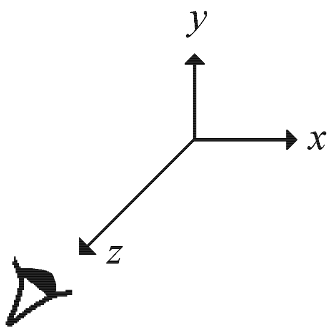

Fig. 1. Coordinate system most frequently used in optics. The observer looks towards the source and measures positive as counter-clockwise. This follows from the equation $\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(\omega t - kz)}$ where the wave propagates along the z-axis with time.

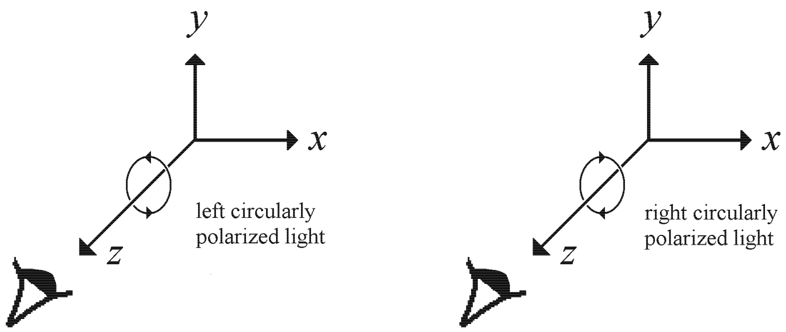

Fig. 2. Diagrams depicting left-circularly polarized light and right-circularly polarized light.

same orientation of FA as $\begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$. These texts all give the Jones vector for the right-circularly polarized light as $\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, while Jaio and Wang [1] give this Jones vector to represent left-circularly polarized light. Hence, the researcher in optical physics must be cautious as to which coordinate system and transformations they put in use. What is usually referred to as right-circularly polarized light in the field of optics is usually regarded as left-circularly polarized light (positive helicity) in quantum physics [6–8].

2. *Derivation of the transformation*

The expression for the Jones matrix of a retarder depends on the expression for the analytical signal of the electromagnetic field. As a convention, the analytical signal of the electromagnetic field is expressed as $\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(\omega t - kz)}$.

A Jones matrix, J, transforms an input Jones vector E_{in} into an output Jones vector E_{out}, while a Mueller matrix, M, transforms an input Stokes vector S_{in} into an output Stokes vector S_{out}:

$$E_{\mathrm{out}} = J E_{\mathrm{in}} = \begin{bmatrix} J_{00} & J_{01} \\ J_{10} & J_{11} \end{bmatrix} E_{\mathrm{in}} \,, \tag{1}$$

and

$$S_{\mathrm{out}} = M S_{\mathrm{in}} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{bmatrix} S_{\mathrm{in}} \,. \tag{2}$$

Here, the Jones matrix is a 2×2 complex valued matrix with 4 components and the Mueller matrix is a 4×4 real valued matrix with 16 components. The Jones vector is a 2×1 column vector, whereas the Stokes vector is a 4×1 column vector. For a polarized beam of radiation, the Jones vector may be written as [9, 10]

$$E = \begin{bmatrix} H \\ K\mathrm{e}^{\mathrm{i}\theta} \end{bmatrix} , \tag{3}$$

and its Hermitian conjugate, E^* of E is given by $\begin{bmatrix} H & K\mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix}$. Thus

$$E^* E = H^2 + K^2. \tag{4}$$

This is the first of the Stokes parameters of the beam which is represented by S_0. If we define the matrix σ_0 to be $\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1}$, then we get

$$E^* \sigma_0 E = H^2 + K^2 = S_0 \,.$$

Similarly,

$$E^*\sigma_1 E = H^2 - K^2 = S_1\,,$$

$$E^*\sigma_2 E = 2HK\cos\theta = S_2\,,$$

$$E^*\sigma_3 E = 2HK\sin\theta = S_3\,,$$

where

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}.$$

Here, σ_0 is the identity matrix, while σ_1, σ_2 and σ_3 can be considered as the "optical" Pauli matrices. These matrices provide a basis for the determination of the Mueller matrix. Notice that these matrices differ from those used in quantum mechanics where the roles of σ_2 and σ_3 are interchanged [7−8]. These matrices satisfy the following relation

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k\,, \quad i,\,j,\,k = 1,2,3\,, \tag{5}$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{6}$$

is the Kronecker delta, and

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any index is equal to any other index} \\ +1 & \text{if } i,\, j,\, k \text{ form an even permutation of 1, 2, 3} \\ -1 & \text{if } i,\, j,\, k \text{ form an odd permutation of 1, 2, 3} \end{cases}$$

is the Levi-Civita tensor.

Given the basis matrices $\{\sigma_\mu\}$, the basis vectors $\{s_\mu\}$ associated to the basis matrices $\{\sigma_\mu\}$ are defined as follows [11]: For a given $\mu = 0$, 1, 2, 3, define the four components $[s_\mu]_\chi = [\sigma_\mu]_{\gamma\delta}$, $\chi = 2\gamma + \delta$. Hence, for $\mu = 0$,

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} [s_0]_0 & [s_0]_1 \\ [s_0]_2 & [s_0]_3 \end{bmatrix}. \tag{7}$$

From this we deduce

$$s_0 = \begin{bmatrix} [s_0]_0 \\ [s_0]_1 \\ [s_0]_2 \\ [s_0]_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

Similarly,

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 \\ -\mathrm{i} \\ \mathrm{i} \\ 0 \end{bmatrix}. \tag{9}$$

Thus the normalized matrix becomes

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\mathrm{i} \\ 0 & 0 & 1 & \mathrm{i} \\ 1 & -1 & 0 & 0 \end{bmatrix}. \tag{10}$$

In component form, the Mueller matrix can be written as

$$\begin{aligned} M_{\mu\nu} &= \frac{1}{2}\mathrm{Tr}\left(\sigma_\mu J \sigma_\nu^* J^\dagger\right) = \frac{1}{2}\left[\sigma_\mu\right]_{kl} J_{lm} \left[\sigma_\nu^*\right]_{mn} J_{nk}^\dagger \\ &= \frac{1}{2} J_{lm} J_{kn}^* \left[\sigma_\mu\right]_{kl} \left[\sigma_\nu^*\right]_{mn} = \frac{1}{2}\left(J \otimes J^*\right)_{lk,mn} \left[\sigma_\mu\right]_{kl} \left[\sigma_\nu^*\right]_{mn} \\ &= \frac{1}{2}\left[\sigma_\mu^*\right]_{lk} \left(J \otimes J^*\right)_{lk,mn} \left[\sigma_\nu^*\right]_{mn} = \frac{1}{2}\left[\sigma_\mu^*\right]_{\alpha} \left(J \otimes J^*\right)_{\alpha\beta} \left[\sigma_\nu^*\right]_{\beta}, \end{aligned} \tag{11}$$

where $\otimes$ represents the Kronecker or direct product and $\dagger$ denotes the conjugate transpose of matrices.

Hence,

$$M_{\mu\nu} = S_{\alpha\mu}^* \left(J \otimes J^*\right)_{\alpha\beta} S_{\beta\nu}^* = S_{\mu\alpha}^\dagger \left(J \otimes J^*\right)_{\alpha\beta} S_{\beta\nu}^* = \left[S^\dagger \left(J \otimes J^*\right) S^*\right]_{\mu\nu}, \tag{12}$$

i.e.,

$$M = S^{*T} \left(J \otimes J^*\right) S^*,$$

where T represents the matrix transpose. Equivalently,

$$M = \left(S^*\right)^{-1} \left(J \otimes J^*\right) S^*.$$

Letting $U = S^\dagger = S^{*T} = \left(S^*\right)^{-1}$, we get

$$M = U\left(J \otimes J^*\right) U^{-1}, \tag{13}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \end{bmatrix}. \tag{14}$$

Thus the Mueller matrix is formed by using a unitary transformation with the Kronecker product of J and J^*. This transformation matrix provides a change of basis from the "optical" Pauli basis to the standard basis.

As a check, consider the Jones matrix for a quarter wave plate, with fast axis horizontal [2, 3]

$$J = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

$$J \otimes J^* = \begin{bmatrix} 1\big[J^*\big] & 0\big[J^*\big] \\ 0\big[J^*\big] & \mathrm{i}\big[J^*\big] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and since $M = U\big(J \otimes J^*\big)U^{-1}$, then

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i} \\ 0 & 0 & 1 & -\mathrm{i} \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{15}$$

which is the required Mueller matrix for a quarter wave plate, fast axis horizontal as given in Refs. [2] and [12].

Note that the unitary matrix U defined here in this paper is the complex conjugate of the matrix defined in Refs. [1], [11], [13] and [14]. In these references, the matrix U is defined as

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & \mathrm{i} & -\mathrm{i} & 0 \end{bmatrix}.$$

Using this matrix, we carry out the above calculation for the quarter wave plate to get

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & \mathrm{i} & -\mathrm{i} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\mathrm{i} \\ 0 & 0 & 1 & \mathrm{i} \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which gives the Mueller matrix for a quarter wave plate with fast axis vertical.

Given the Jones vector E, we can transform to the corresponding Stokes vector, V, by using the transformation

$$V = \sqrt{2}\, U\big(E \otimes E^*\big) \,. \tag{16}$$

For example, given the normalized Jones vector for left-circularly polarized light (LCPL) as $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}$,

$$E \otimes E^* = \frac{1}{2} \begin{pmatrix} 1 \\ -\mathrm{i} \\ \mathrm{i} \\ 1 \end{pmatrix}$$

and

$$V = \sqrt{2} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \end{bmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ -\mathrm{i} \\ \mathrm{i} \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{17}$$

which gives the Stokes vector for the left-circularly polarized light.

3. Conclusion

An algebraic method of developing the Mueller matrix from the Jones matrix has been presented. This method transforms the complex-valued Jones matrix into the corresponding real-valued Mueller matrix. Instead of measuring all sixteen components of a Mueller matrix, the four components of the Jones matrix can be measured and transformed into components of the Mueller matrix. This could be of considerable importance in optical physics, biophysics and radio interferometry where it is desired to measure and analyze the Mueller matrix of a sample or some radiating device.

References

[1] S. Jaio and L. V. Wang, J. Biomed. Optics **7**, 3 (2002) 350 – 358.

[2] E. Hecht, *Optics*, 4th ed., Addison Wesley (2002) p. 378.

[3] F. Pedrotti and L. Pedrotti, *Introduction to Optics*, 2nd ed., Prentice Hall (1993) p. 288.

[4] G. Fowles, *Introduction to Modern Optics*, Holt, Rinehart, Winston (1968) p. 45.

[5] J. Meyer-Arendt, *Introduction to Classical and Modern Optics*, 4th ed., Prentice Hall (1995) p. 297.

[6] J. D. Jackson, *Classical Electrodynamics*, 2nd ed., Wiley (1975) p. 274 – 275.

[7] D. Park, *Intoduction to the Quantum Theory*, 3rd ed., McGraw Hill (1992).

[8] R. L. Liboff, *Introductory Quantum Mechanics*, 3rd ed., Addison-Wesley (1998) p. 574 – 576.

[9] A. Gerrard and J. M Burch, *Introduction to Matrix Methods in Optics*, Wiley (1975) p. 211.

[10] R. Clark Jones, *A new Calculus for the Treatment of Optical Systems*, J. Opt. Soc. Am. **31** (1941) 488.

[11] A. Aiello and J. P. Woerdman, *Linear Algebra for Mueller Calculus*, arXiv:math-ph/0412061.

[12] W. S. Bickel and W. M. Bailey, *Stokes vectors*, Am. J. Phys. **53** (5) (1985).

[13] J. Byrne, J. Phys. B: Atom. Molec. Phys. **4** (1971) 945.

[14] Sudha and A. V. Gopala Rao, *Polarization Elements – A group Theoretical Study*, arXiv:physics/0007079 (2000).

JONES-MUELLEROVA PREOBRAZBA

Jednostavnom preobrazbom moguće je izvesti realnu Muellerovu matricu iz kompleksne Jonesove matrice. Preobrazba se provjerava prevođenjem Jonesovih matrica za optičke naprave u odgovarajuće Muellerove matrice. Izvodi se i Stokesov vektor iz odgovarajućeg Jonesovog vektora.