Printed ISSN 1330–0008
Online ISSN 1333–9125
CD ISSN 1333–8390
CODEN FIZAE4

ACTION PRINCIPLE FOR A KIND OF MECHANICAL SYSTEM WITH FRICTION FORCE

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Received 20 June 2006; Revised manuscript received 5 August 2008
Accepted 3 September 2008 Online 25 September 2008

A least action principle for mechanical systems with friction was obtained. Some action principles, such as El-Nabulsi's fractional action-like variational approach, are given as the special cases.

PACS numbers: 02.30.Xx, 46.40.-f UDC 531.553, 534.112
Keywords: action principle, nonconservative system

1. Introduction

It is well known that the least action principle has played an important role in physics. From the mathematical view of point, the existence of an action principle depends on the self-adjoint property of the corresponding Lagrangian. For example, if we consider a string system with a friction force which depends on the velocity linearly, then we can't construct an appropriate Lagrangian by adding a related term to give an action principle [1]. Recently, El-Nabulsi [2] applied fractional integral to define an action as

$$S = I^{\alpha}(L) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} L(q(\tau), q'(\tau))(t-\tau)^{\alpha-1} \mathrm{d}\tau \,, \tag{1}$$

and obtained the corresponding Euler-Lagrange equation and Hamilton's equations. Using his fraction action-like variational approach, El-Nabulsi studied some cosmology problems and got some results [3]. However, El-Nabulsi's action is too special to obtain the motion equation to the above mentioned of string system. In the present letter, we will propose a general action to give the Euler-Lagrange equation and Hamilton's equations. As a result, El-Nabulsi's fractional action is only a special case of our action.

2. Action principle

We take the action as

$$S = \int_{t_0}^{t} L(q'(t), q(t))\rho(t)\mathrm{d}t \,, \tag{1}$$

where $L(q'(t), q(t))$ is the Lagrangian without friction force. $\rho(t)$ is a differentiable function whose form can be determined according to the concrete model. We now derive the Euler-Lagrange equation from this action. Taking the variation to S, we have

$$\delta S = \int_{t_0}^{t} \left\{ L(q'(t) + (\delta q)'(t), q(t) + \delta q(t)) - L(q'(t), q(t)) \right\} \rho(t)\mathrm{d}t$$

$$= \int_{t_0}^{t} \rho(t) \left\{ \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q'}\right) - \frac{\rho'(t)}{\rho(t)} \frac{\partial L}{\partial q'} \right\} \delta q \mathrm{d}t \,. \tag{3}$$

So we obtain the following Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q'}\right) - \frac{\rho'(t)}{\rho(t)} \frac{\partial L}{\partial q'} = 0 \,. \tag{4}$$

Of course the above Euler-Lagrange equation can be derived by another direct way. In fact, we take a Lagrangian $L_1 = \rho(t)L$, and the corresponding Euler-Lagrange equation follows from the ordinary Euler-Lagrange equation

$$\frac{\partial L_1}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial q'}\right). \tag{5}$$

For example, if we take $\rho(t) = \exp(\varepsilon t)$, then the corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q'}\right) - \varepsilon \frac{\partial L}{\partial q'} = 0 \,. \tag{6}$$

For a string system with a friction force $f(t) = \varepsilon x'(t)$, from Eq.(6), we have its equation of motion

$$x''(t) + \varepsilon x'(t) + \frac{k}{m} x(t) = 0 \,, \tag{7}$$

where k and m are the elastic coefficient and mass of the string, respectively, and ε the coefficient of friction.

If we take $\rho(t) = \exp(\frac{1}{2}\varepsilon t^2)$, then the equation of motion is

$$x''(t) + \varepsilon t x'(t) + \frac{k}{m}\, x(t) = 0\,. \tag{8}$$

Then we obtain as the result the Emden-Fauller equation given in Ref. [4].

If we take $\rho = \varepsilon t^2$, then the equation of motion is

$$x''(t) + \frac{2\varepsilon}{t} x'(t) + \frac{k}{m}\, x(t) = 0\,. \tag{9}$$

If we take $\rho(t) = \dfrac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$, we obtain the corresponding results of El-Nabulsi [1].

Remark. If we take $\rho = 1$, then we get the classical action principle.

3. *Hamilton's canonical form*

Hamilton function is defined by $H = pq' - L$, where $p = \partial L/\partial q'$. Then

$$\mathrm{d}H = \frac{\partial H}{\partial p}\mathrm{d}p + \frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial t}\mathrm{d}t\,, \tag{10}$$

and

$$\mathrm{d}H = q'\mathrm{d}p - \frac{\partial L}{\partial q}\mathrm{d}q - \frac{\partial L}{\partial t}\mathrm{d}t\,, \tag{11}$$

so we have

$$q' = \frac{\partial H}{\partial p}, \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{12}$$

According to the Euler-Lagrange Eq. (4), we have

$$p' = \frac{\partial L}{\partial q} - \frac{\rho'(t)}{\rho(t)}\, p\,. \tag{13}$$

From the above equations, we have the following result

Theorem 1: Euler-Lagrange equations are equivalent to the following Hamilton equations:

$$p' = -\frac{\partial H}{\partial q} - \frac{\rho'(t)}{\rho(t)}\, p\,, \qquad q' = \frac{\partial H}{\partial p}\,. \tag{14}$$

For the above string system, we have the corresponding Hamilton equations:

$$p' = -kq - \varepsilon p, \qquad q' = \frac{p}{m}\,. \tag{15}$$

4. Another method

We take a new time variable $s = \int\limits_0^t \rho(\tau)\mathrm{d}\tau$. Then we have $\mathrm{d}s/\mathrm{d}t = \rho(t)$, $\mathrm{d}q/\mathrm{d}t = \rho(t)[\mathrm{d}q/\mathrm{d}s]$ and $\mathrm{d}^2q/\mathrm{d}t^2 = \rho^2(t)[\mathrm{d}^2q/\mathrm{d}s^2] + \rho'(t)[\mathrm{d}q/\mathrm{d}s]$.

If we consider the new time system, we have $L = L(\rho(t)q'(s), q(s))$, then action becomes

$$S(q) = \int\limits_{s_0}^{s_1} L(\rho(t)q'(s), q(s))\mathrm{d}s\,. \tag{16}$$

For this action, the corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial q(s)} = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q'(s)}\right). \tag{17}$$

Theorem 2: Euler-Lagrange equations (17) and (4) are equivalent.

Proof: We only need to derive Eq. (4) from Eq. (17). Since $q'(t)$ is included in L only by the quadratic form, so we have

$$\frac{\partial L}{\partial q'(t)}\Big|_{t=s} = \frac{1}{\rho(t)}\frac{\partial L}{\partial q'(t)}.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q'(s)}\right) = 2\rho'(t)\frac{\partial L}{\partial q'(t)}\Big|_{t=s} + \rho(t)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial q'(t)}\Big|_{t=s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q'}\right) + \frac{\rho'(t)}{\rho(t)}\frac{\partial L}{\partial q'}. \tag{18}$$

The proof is completed.

In fact, if we take the action as

$$S(q) = \int\limits_{s_0}^{s_1} L(\sigma(s)q'(s), q(s))\mathrm{d}s\,, \tag{19}$$

and assume $t = \int\limits_0^s \frac{1}{\sigma(s)}$ and $\sigma(s) = \rho(t)$, then we have the same results as derived from the action (2).

5. Continuous case

In the case of a continuous field, we take the action as

$$S = \int\limits_{t_0}^{t_1}\int \rho(x,t)L(\partial_x\varphi, \partial_t\varphi, \varphi)\mathrm{d}x\mathrm{d}t\,. \tag{20}$$

The corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial \varphi} - \partial_x \left(\frac{\partial L}{\partial \varphi_x} \right) - \partial_t \left(\frac{\partial L}{\partial \varphi_t} \right) = \frac{\rho_x}{\rho} \left(\frac{\partial L}{\partial \varphi_x} \right) + \frac{\rho_t}{\rho} \left(\frac{\partial L}{\partial \varphi_t} \right). \tag{21}$$

For example, if we take $L = \frac{1}{2}(u_t)^2 - \frac{1}{2}a^2(u_x)^2$ and $\rho = \exp(\varepsilon x + \eta t)$, then we have

$$u_{tt} - a^2 u_{xx} + \varepsilon u_x + \eta u_t = 0\,. \tag{22}$$

6. Friction geometry

We consider a Riemman space M with a Riemman metric g, then the local metric is presented by $\mathrm{d}s = g_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j$. We take the action functional as

$$S = \int_{t_0}^{t} \sqrt{g_{ij}(x)(x^i(\tau))'(x^j(\tau))'}\ \rho(\tau)\mathrm{d}\tau\,. \tag{23}$$

The Euler-Lagrange equation is

$$(x^k)'' + \frac{\rho'}{\rho}(x^k)' + \Gamma^k_{ij}\ (x^i)'(x^j)' = 0\,, \tag{24}$$

where Γ^k_{ij} are the Christoffel coefficients. Eq. (24) is the modified geodesic equation. Therefore, we can deal with the motion of a particle with a friction force from a view of geometry.

7. Conclusion and discussion

Through multiplying a time factor in the free Lagrangian, we give a general action principle from which the equation of motion of some systems with a friction force are given. As a result, some other action principles, such as El-Nabulsi's fractional action-like variational approach, are given as the special cases. Of course, we must point out that we need a physical explanation to the proposed action principle.

Acknowledgements

Thanks to Dr. Hua-Tong Yang (Peking University) for fruitful discussions.

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NAČELO DJELOVANJA ZA VRSTU MEHANIČKIH SUSTAVA S TRENJEM

Izvodimo načelo najmanjeg djelovanja za mehanički sustav s trenjem. Neka načela djelovanja, kao El-Nabulsijev razlomčani varijacijski pristup djelovanja, su posebni slučajevi.