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LORENTZ AND “APPARENT” TRANSFORMATIONS OF THE ELECTRIC AND MAGNETIC FIELDS

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It has recently been proved in the tensor formalism and the Clifford, i.e., geometric, algebra formalism that the usual transformations of the three-dimensional (3D) vectors of the electric and magnetic fields differ from the Lorentz transformations (boosts) of the corresponding 4D quantities that represent the electric and magnetic fields. In this paper, using geometric algebra formalism, this fundamental difference is examined representing the electric and magnetic fields by bivectors and 1-vectors.

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1. *Introduction*

Recently [1-3] it has been proved that, contrary to the general belief, the usual transformations of the three-dimensional (3D) vectors of the electric and magnetic fields, see, e.g., Jackson's well-known textbook [4], Eqs. (11.148) and (11.149), differ from the Lorentz transformations (LT) (boosts) of the corresponding 4D quantities that represent the electric and magnetic fields. The usual transformations will be called the “apparent” transformations (AT) and the name will be explained in Sec. 4. This new approach $[1-3, 5, 6]$ with 4D geometric quantities always agrees with the principle of relativity and with experiments. This is shown by comparison with experiments, e.g., the motional emf [2], the Faraday disk [3] and the Trouton-Noble experiment $[5, 6]$. On the other hand, this is not the case with the usual approach in which the electric and magnetic fields are represented by the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ that transform according to the AT. The mentioned agreement with experiments is independent of the chosen reference frame and of the chosen system of coordinates in it. The main point in the geometric approach $[1-3, 5, 6]$ is that the physical meaning, both theoretically and *experimentally*, is attributed to the 4D geometric quantities, and not, as usual, to the 3D quantities.

In this paper I shall present a simplified version of the proof of the difference between the LT and the AT that is already given in Secs. 3.3 and 4 in Ref. [2]. The version presented here is better suited for students and teachers than that one in Ref. [2]. For all mathematical details for the used geometric algebra formalism readers can consult Refs. [7, 8] and a brief summary in Sec. 2.

As shown in Refs. [2, 3], the electric and magnetic fields can be represented by different algebraic objects; 1-vectors, bivectors or their combination. The representation with 1-vectors E and B is simpler than others and also closer to the usual expressions with the 3D vectors $\mathbf{E}$ and $\mathbf{B}$, but here we shall consider in more detail the representation with bivectors, while only briefly that one with 1-vectors. The reason is that the representation with bivectors, as in our Eq. (2), is employed in Refs. [7, 8] and we want to make comparison with their results. In Subsection 3.1, the Lorentz invariant representation, E_v and B_v, is presented as introduced in Refs. [2] and [3]. In Subsection 3.2, from these E_v and B_v, we simply derive the observer-dependent expressions for the electric $\mathbf{E}_H$ and magnetic $\mathbf{B}_H$ fields, which are solely exploited in Refs. [7, 8]. $\mathbf{E}'_H$ (and $\mathbf{B}'_H$), which are the LT (the active ones) of $\mathbf{E}_H$ (and $\mathbf{B}_H$), Eqs. (10) and (11), are derived in Sec. 4 using the fact that every multivector must transform under the active LT in the same way, i.e., according to Eq. (9). Furthermore, it is known that any multivector, when written in terms of components and a basis, must remain unchanged under the passive LT, like some general bivector in Eq. (14). Hence observers in relatively moving inertial frames S and S' will "see" the same 4D geometric quantity, i.e., as in Eq. (15) for $\mathbf{E}_H$. These fundamental achievements for the LT of bivectors $\mathbf{E}_H$ (and $\mathbf{B}_H$) were first obtained in Ref. [2]. Hestenes [7] and the Cambridge group [8] derived the transformations for $\mathbf{E}_H$ and $\mathbf{B}_H$ (Ref. [7], *Space-Time Algebra,* Eq. (18.22), *New Foundations for Classical Mechanics,* Ch. 9, Eqs. (3.51a,b) and Ref. [8], Sec. 7.1.2, Eq. (7.33)) in the way that is presented in Sec. 5, Eqs. (16) and (17) for $\mathbf{E}'_{H,at}$, and Eqs. (18), (19) and (20) for the components. The transformations for components, Eqs. (19) and (20), are identical to the usual transformations for components of the 3D $\mathbf{E}$ and $\mathbf{B}$, Ref. [4], Eq. (11.148). Such usual transformations are quoted in every textbook and paper on relativistic electrodynamics already from the time of Einstein's fundamental paper [9], and Lorentz's [10] and Poincaré's [11, 12] papers. They are always considered (including Refs. [7, 8]) to be the LT of the electric and magnetic fields. However, it is obvious from (16) and (17) that $\mathbf{E}'_{H,at}$ is not obtained by the active LT from $\mathbf{E}_H$, since (16) is drastically different than the correct LT (9) and (10). Furthermore, as seen from Eq. (21), the relation (14) is not fulfilled, which means that $\mathbf{E}_H$ and $\mathbf{E}'_{H,at}$ are not the same physical quantity for relatively moving observers in S and S'. Again, completely different result than the one obtained by the correct passive LT, Eq. (15). This shows that neither the usual transformations of the electric and magnetic fields from Refs. [7, 8] nor the usual transformations for components, Ref. [4], Eqs. (11.148), are the LT. In Section 6, a short presentation of the fundamental difference between the LT and the AT when dealing with 1-vectors E and B is given. The conclusions are given in Sec. 7. Recently, the existence of a fundamental difference between the LT and the AT is once again simply proved and used in the derivations and explanations (tensor formalism, with tensors as 4D geometric quantities) in my paper in Phys. Rev. Lett., Ref. [13].

2. A brief summary of geometric algebra

Here, for readers' convenience, we provide a brief summary of the geometric algebra. Usually Clifford vectors are written in lower case (a) and general multivectors (Clifford aggregates) in upper case (A). The space of multivectors is graded and multivectors containing elements of a single grade, r, are termed homogeneous and usually written A_r. The geometric (Clifford) product is written by simply juxtaposing multivectors AB. A basic operation on multivectors is the degree projection $\langle A\rangle_r$ which selects from the multivector A its $r-$vector part ($0 =$ scalar, $1 =$ vector, $2 =$ bivector ....). The geometric product of a grade-r multivector A_r with a grade-s multivector B_s decomposes into $A_r B_s = \langle AB\rangle_{r+s} + \langle AB\rangle_{r+s-2} ... + \langle AB\rangle_{|r-s|}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms, respectively, of the above series; $A_r \cdot B_s \equiv \langle AB\rangle_{|r-s|}$ and $A_r \wedge B_s \equiv \langle AB\rangle_{r+s}$. For vectors a and b we have: $ab = a\cdot b + a\wedge b$, where $a\cdot b \equiv (1/2)(ab+ba)$, $a\wedge b \equiv (1/2)(ab-ba)$.

In this paper, the notation will not be the same as in the above mathematical presentation. Namely some 1-vectors will be denoted in lower case, like v (the velocity), x (the position 1-vector), while some others in upper case, like the 1-vectors of the electric and magnetic fields E and B, respectively (in Sec. 5). Bivectors will be denoted in upper case but without subscript that denotes the grade. Thus the electromagnetic field F, the electric and magnetic fields, E_v and B_v, $\mathbf{E}_H$ and $\mathbf{B}_H$, are all bivectors.

3. Electric and magnetic fields as bivectors

In the geometric approach used in this paper physical quantities will be represented by 4D geometric quantities, multivectors, that are defined without reference frames, or, when some basis has been introduced, these quantities are represented as 4D geometric quantities comprising both components and a basis. Such 4D geometric quantities that are defined without reference frames will be called the absolute quantities (AQs), while their representations in some basis will be called coordinate-based geometric quantities (CBGQs).

For example, in Refs. [7, 8], one introduces the standard basis. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors $\{\gamma_\mu\}, \mu = 0...3$, satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+---)$. This basis, the standard basis, is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone. The γ_k ($k = 1, 2, 3$) are spacelike vectors. This algebra is often called the Dirac algebra D and the elements of D are called $d-$numbers. The basis vectors γ_μ generate by multiplication a complete basis for the spacetime algebra: 1, γ_μ, $\gamma_\mu \wedge \gamma_\nu$, $\gamma_\mu \gamma_5$, γ_5 (16 independent elements). γ_5 is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra. It is worth noting that the standard basis $\{\gamma_\mu\}$ corresponds, in fact, to the specific system of coordinates, i.e., to Einstein's system of coordinates. In the Einstein system of coordinates, the Einstein synchronization [9] of distant clocks and Cartesian space coordinates x^i are used in the chosen inertial frame of reference. However, different systems of coordinates of an inertial frame of reference are allowed and they are all equivalent in the description of physical phenomena.

For simplicity and for easier understanding we shall also deal only with the standard basis, but remembering that the approach with 4D geometric quantities holds for any choice of basis in M^4.

3.1. Lorentz invariant electric and magnetic fields

The electromagnetic field is represented by a bivector-valued function $F = F(x)$ in the spacetime. As shown in Refs. [2, 3], the observer-independent F can be decomposed into two bivectors E_v and B_v representing the electric and magnetic fields and the unit time-like 1-vector v/c as

$$\begin{aligned} F &= E_v + cIB_v, \qquad E_v = (1/c^2)(F \cdot v) \wedge v = (1/2c^2)(F - vFv), \\ IB_v &= (1/c^3)(F \wedge v) \cdot v = (1/2c^3)(F + vFv), \end{aligned} \tag{1}$$

where I is the unit pseudoscalar and v is the velocity (1-vector) of a family of observers who measures E_v and B_v fields. Observe that E_v and B_v depend not only on F but on v as well. All quantities F, E_v, B_v, I and v are defined without reference frames, i.e., they are AQs (I is defined algebraically without introducing any reference frame, as in Ref. [14] Sec. 1.2.) Their representations in some basis are CBGQs. For example, in the $\{\gamma_\mu\}$ basis the AQ E_v from (1) is represented by the following CBGQ, $E_v = (1/c^2)F^{\mu\nu}v_\nu v^\beta \gamma_\mu \wedge \gamma_\beta$

3.2. Electric and magnetic fields in the γ_0-frame

For comparison with the usual treatments [7,8], let us choose the frame in which the observers who measure E_v and B_v are at rest. For them $v = c\gamma_0$. This frame will be called the frame of "fiducial" observers or the γ_0-frame. In that frame E_v and B_v from (1) become the observer dependent (γ_0-dependent) $\mathbf{E}_H$ and $\mathbf{B}_H$ and instead of Eq. (1), we have

$$\begin{aligned} F = \mathbf{E}_H + c\gamma_5 \mathbf{B}_H, \quad \mathbf{E}_H = (F \cdot \gamma_0)\gamma_0 = (1/2)(F - \gamma_0 F \gamma_0), \\ \gamma_5 \mathbf{B}_H = (1/c)(F \wedge \gamma_0)\gamma_0 = (1/2c)(F + \gamma_0 F \gamma_0). \end{aligned} \tag{2}$$

(The subscript H is for “Hestenes.”) E_v and B_v in the γ_0-frame are denoted as $\mathbf{E}_H$ and $\mathbf{B}_H$ since they are identical to 4D quantities used by Hestenes [7] and the Cambridge group [8] for the representation of the electric and magnetic fields. We note that such procedure was never used by Hestenes [7] and the Cambridge group [8] since they deal from the outset only with γ_0 and thus with a space-time split in the γ_0-frame, i.e., with the relations (2). This shows that the space-time split and the corresponding observer-dependent form for the electric and magnetic fields, (2), which is used in Refs. [7, 8], is simply obtained in our approach going to the frame of the “fiducial” observers, i.e., replacing some general velocity v in (1) by $c\gamma_0$.

$\mathbf{E}_H$ and $\mathbf{B}_H$ from (2) can be written as CBGQs in the standard basis $\{\gamma_\mu\}$. They are

$$\mathbf{E}_H = F^{i0}\gamma_i \wedge \gamma_0, \quad \mathbf{B}_H = (1/2c)\varepsilon^{kli0}F_{kl}\gamma_i \wedge \gamma_0. \tag{3}$$

It follows from (3) that the components of $\mathbf{E}_H$, $\mathbf{B}_H$ in the $\{\gamma_\mu\}$ basis (i.e., in the Einstein system of coordinates) give rise to the tensor (components) $(\mathbf{E}_H)^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot \mathbf{E}_H) = (\gamma^\nu \wedge \gamma^\mu) \cdot \mathbf{E}_H$, (and the same for $(\mathbf{B}_H)^{\mu\nu}$) which, written out as a matrix, have entries

$$\begin{aligned} (\boldsymbol{E}_H)^{i0} &= F^{i0} = E^i, \quad (\boldsymbol{E}_H)^{ij} = 0, \\ (\boldsymbol{B}_H)^{i0} &= (1/2c)\varepsilon^{kli0}F_{kl} = B^i, \quad (\boldsymbol{B}_H)^{ij} = 0. \end{aligned} \tag{4}$$

$(\mathbf{E}_H)^{\mu\nu}$ is antisymmetric, i.e., $(\mathbf{E}_H)^{\nu\mu} = -(\mathbf{E}_H)^{\mu\nu}$, and the same holds for $(\mathbf{B}_H)^{\mu\nu}$. $(\mathbf{E}_H)^{\mu\nu}$ from Eq. (4) can be written in a matrix form as

$$(\mathbf{E}_H)^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 = F^{10} & 0 & 0 & 0 \\ E^2 = F^{20} & 0 & 0 & 0 \\ E^3 = F^{30} & 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

and readers can check that the same matrix form is obtained for $(\mathbf{B}_H)^{\mu\nu}$. ($(\mathbf{B}_H)^{10} = (1/c)F^{32} = B^1$.)

Thus from Eqs. (3) and (4) or (5), we see that both bivectors $\mathbf{E}_H$ and $\mathbf{B}_H$ are parallel to γ_0,

$$\mathbf{E}_H \wedge \gamma_0 = \mathbf{B}_H \wedge \gamma_0 = 0\,, \tag{6a}$$

and consequently all space-space components of $(\mathbf{E}_H)^{\mu\nu}$ and $(\mathbf{B}_H)^{\mu\nu}$ are zero,

$$(\mathbf{E}_H)^{ij} = (\mathbf{B}_H)^{ij} = 0\,. \tag{6b}$$

In the usual covariant approaches [4], the components of the 3D $\mathbf{E}$ and $\mathbf{B}$ are identified with six independent components of $F^{\mu\nu}$ according to the relations

$$E_i = F^{i0}, \quad B_i = (-1/2c)\varepsilon_{ikl}F_{kl}. \tag{7}$$

Such an identification was first given in Einstein's fundamental paper on general relativity [15]. In (7) and hereafter the components of the 3D fields $\mathbf{E}$ and $\mathbf{B}$ are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric ε tensor too. The super- and subscripts are used only on the components of the 4D quantities.

Comparing (4) and (7), we see that they similarly identify the components of the electric and magnetic fields with six independent components of $F^{\mu\nu}$. However, there are important differences between the relations (3), (4) or (5), and (7). In the usual covariant approaches, e.g., [4], the 3D $\mathbf{E}$ and $\mathbf{B}$, as *geometric quantities in the 3D space*, are constructed from these six independent components of $F^{\mu\nu}$ and *the unit 3D vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, e.g., $\mathbf{E} = F^{10}\mathbf{i} + F^{20}\mathbf{j} + F^{30}\mathbf{k}$. Observe that the mapping, i.e., the simple identification, Eq. (7), of the components E_i and B_i with some components of $F^{\mu\nu}$ (defined on the 4D spacetime) is not a permissible tensor operation, i.e., it is not a mathematically correct procedure. (A permissible tensor operation with components of some tensors produces components of a new tensor, e.g., the summation of components of two tensors gives the components of the sum of two tensors.) The same holds for the construction of the *3D vectors* $\mathbf{E}$ and $\mathbf{B}$ in which the components of the *4D quantity* $F^{\mu\nu}$ are multiplied with *the unit 3D vectors,* see Ref. [3] for the more detailed discussion. On the other hand, as seen from Eqs. (3), (4) or (5), $\mathbf{E}_H$ and $\mathbf{B}_H$ and their components $(\mathbf{E}_H)^{\mu\nu}$ and $(\mathbf{B}_H)^{\mu\nu}$ are obtained by a correct mathematical procedure from the 4D geometric quantities F and γ^μ. The components $(\mathbf{E}_H)^{\mu\nu}$ and $(\mathbf{B}_H)^{\mu\nu}$ are multiplied by the unit bivectors $\gamma_i \wedge \gamma_0$ (4D quantities) to form the geometric 4D quantities $\mathbf{E}_H$ and $\mathbf{B}_H$. In such a treatment, the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, (geometric quantities in the *3D space*) do not appear at any point.

Furthermore, it is worth noting that $F^{\mu\nu}$ are only components (numbers) that are (implicitly) determined in Einstein's system of coordinates. Components are frame-dependent numbers (frame-dependent because the basis refers to a specific frame). Components tell only part of the story, while the basis contains the rest of the information about the considered physical quantity. These facts are completely overlooked in all usual covariant approaches and in the above identifications (7) of E_i and B_i with some components of $F^{\mu\nu}$.

4. *Lorentz transformations of electric and magnetic fields as bivectors*

Let us now apply the active LT (only boosts are considered) to $\mathbf{E}_H$ and $\mathbf{B}_H$ from Eq. (3). In the usual geometric algebra formalism [7, 8], the LT are considered as active transformations; the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis $\{\gamma_\mu\}$) are transformed into the components of a new 1-vector relative to the same frame (the basis $\{\gamma_\mu\}$ is not changed). Furthermore, the LT are described with rotors R, $R\widetilde{R} = 1$, in the usual way as $p \to p' = Rp\widetilde{R} = p'_\mu \gamma^\mu$. Remember that the reverse $\widetilde{R}$ is defined by the operation of reversion according to which $\widetilde{AB} = \widetilde{B}\widetilde{A}$, $\widetilde{a} = a$, for any vector a, and it reverses the order of vectors in any given expression. Every rotor in spacetime can be written in terms of a bivector as $R = e^{\theta/2}$. For boosts in arbitrary direction the rotor R is

$$R = e^{\theta/2} = (1 + \gamma + \gamma\beta\gamma_0 n)/(2(1+\gamma))^{1/2}, \tag{8}$$

$\theta = \alpha\gamma_0 n$, β is the scalar velocity in units of c, $\gamma = (1-\beta^2)^{-1/2}$, or in terms of an 'angle' α we have $\tanh\alpha = \beta$, $\cosh\alpha = \gamma$, $\sinh\alpha = \beta\gamma$, and n is not the basis vector but any unit space-like vector orthogonal to γ_0; $e^\theta = \cosh\alpha + \gamma_0 n \sinh\alpha$. One can also express the relationship between two relatively moving frames S and S' in terms of the rotor as $\gamma'_\mu = R\gamma_\mu\widetilde{R}$. For boosts in the direction γ_1, the rotor R is given by the relation (8) with γ_1 replacing n (all in the standard basis $\{\gamma_\mu\}$). For simplicity we shall only consider boosts in the direction γ_1.

As written in Sec. IV in Hestenes' paper [7] in AJP, Lorentz rotations preserve the geometric product. This implies that any multivector M transforms by the active LT in the same way as mentioned above for the 1-vector p, i.e.,

$$M \to M' = RM\widetilde{R}, \tag{9}$$

see, e.g., Eq. (69) in Hestenes' paper [7] in AJP. M in Eq. (9) can be a simple blade or a Clifford aggregate. Also it can be a function of some other multivectors.

Hence, according to (9), under the active LT $\mathbf{E}_H$ from (2) must transform in the following way

$$\mathbf{E}'_H = R[(1/2)(F - \gamma_0 F \gamma_0)]\widetilde{R} = (1/2)[F' - \gamma'_0 F' \gamma'_0] = (F' \cdot \gamma'_0)\gamma'_0, \tag{10}$$

where $F' = RF\widetilde{R}$ and $\gamma'_0 = R\gamma_0\widetilde{R}$. However, as will be shown in Sec. 4, it is surprising that neither Hestenes [7] nor the Cambridge group [8] transform $\mathbf{E}_H$ in the way in which all other multivectors are transformed, i.e., according to (9) and (10).

When the active LT are applied to $\mathbf{E}_H$ from (3), thus when $\mathbf{E}_H$ is written as a CBGQ, then $\mathbf{E}'_H$ becomes

$$\mathbf{E}'_H = R[E^i\gamma_i \wedge \gamma_0]\widetilde{R} = E^1\gamma_1 \wedge \gamma_0 + \gamma(E^2\gamma_2 \wedge \gamma_0$$
$$+ E^3\gamma_3 \wedge \gamma_0) - \beta\gamma(E^2\gamma_2 \wedge \gamma_1 + E^3\gamma_3 \wedge \gamma_1). \tag{11}$$

(We have denoted, as in Eq. (4), $E^i = F^{i0}$.) The components $(\mathbf{E}'_H)^{\mu\nu}$ $((\mathbf{E}'_H)^{\nu\mu} = -(\mathbf{E}'_H)^{\mu\nu})$ can be written in a matrix form as

$$(\mathbf{E}'_H)^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -\gamma E^2 & -\gamma E^3 \\ E^1 & 0 & \beta\gamma E^2 & \beta\gamma E^3 \\ \gamma E^2 & -\beta\gamma E^2 & 0 & 0 \\ \gamma E^3 & -\beta\gamma E^3 & 0 & 0 \end{bmatrix}. \tag{12}$$

The same form can be easily found for $\mathbf{B}'_H$ and its components $(\mathbf{B}'_H)^{\mu\nu}$. (This is left to readers.) Eq. (11) is the familiar form for the active LT of a bivector, here $\mathbf{E}_H$, but written as a CBGQ.

(For some general bivector N, the components transform by the LT as the components of a second-rank tensor

$$\begin{aligned} N'^{23} &= N^{23},\ N'^{31} = \gamma(N^{31} - \beta N^{30}),\ N'^{12} = \gamma(N^{12} + \beta N^{20}), \\ N'^{10} &= N^{10},\ N'^{20} = \gamma(N^{20} + \beta N^{12}),\ N'^{30} = \gamma(N^{30} + \beta N^{13}). \end{aligned} \tag{13}$$

From (13) one easily find $(\mathbf{E}'_H)^{\mu\nu}$ (12) taking into account that the components $(\mathbf{E}_H)^{\mu\nu}$ are determined by Eq. (5).)

It is important to note that

(i) $\mathbf{E}'_H$ and $\mathbf{B}'_H$, in contrast to $\mathbf{E}_H$ and $\mathbf{B}_H$ (see Eqs. (6)), are not parallel to γ_0, i.e., both $\mathbf{E}'_H \wedge \gamma_0 \neq 0$ and $\mathbf{B}'_H \wedge \gamma_0 \neq 0$, which means that there are the space-space components, $(\mathbf{E}'_H)^{ij} \neq 0$ and $(\mathbf{B}'_H)^{ij} \neq 0$. Furthermore,

(ii) *the components* $(\mathbf{E}_H)^{\mu\nu}$ $((\mathbf{B}_H)^{\mu\nu})$ *transform upon the active LT again to the components* $(\mathbf{E}'_H)^{\mu\nu}$ $((\mathbf{B}'_H)^{\mu\nu})$; *there is no mixing of components. Under the active LT* $\mathbf{E}_H$ *transforms to* $\mathbf{E}'_H$ *and* $\mathbf{B}_H$ *to* $\mathbf{B}'_H$. Actually, as already said, this is the way in which every bivector transforms under the active LT.

Instead of using the active LT, we can deal with the passive LT. The essential difference relative to the usual covariant picture is the presence of a basis in a CBGQ. The existence of the basis causes that every 4D CBGQ is invariant under the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. This means that such CBGQ represents *the same physical quantity* for relatively moving 4D observers. For some general bivector N, the components transform according to (13), whereas the basis $\gamma'_\mu \wedge \gamma'_\nu$ transforms by the inverse LT giving that the whole N is unchanged

$$N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)N'^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu, \tag{14}$$

where all primed quantities are the Lorentz transforms of the unprimed ones. It can be checked by the use of (5) and (12) that (14) holds for $\mathbf{E}_H$, i.e., that

$$\mathbf{E}_H = (1/2)(\mathbf{E}_H)^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)(\mathbf{E}'_H)^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu, \tag{15}$$

and the same is valid for $\mathbf{B}_H$.

It is worth noting that one can find the expression for E_v as a CBGQ in the S' frame and in the $\{\gamma'_\mu\}$ basis directly from (1). Namely, in the S', frame the "fiducial" observers (that are at rest in the S frame) are moving with velocity v whose components are $v'^\mu = (\gamma c, -\gamma\beta c, 0, 0)$. Of course, for the whole CBGQ v it holds that $v = v'^\mu\gamma'_\mu = v^\mu\gamma_\mu$, where the components v^μ from S are $v^\mu = (c, 0, 0, 0)$. Then E_v, as a CBGQ in S', becomes $E_v = F'^{10}\gamma'_1 \wedge \gamma'_0 + \gamma^2(F'^{20} + \beta F'^{21})\gamma'_2 \wedge \gamma'_0 + \gamma^2(F'^{30} + \beta F'^{31})\gamma'_3 \wedge \gamma'_0 - \beta\gamma^2(F'^{20} + \beta F'^{21})\gamma'_2 \wedge \gamma'_1 - \beta\gamma^2(F'^{30} + \beta F'^{31})\gamma'_3 \wedge \gamma'_1)$. If the components $F'^{\mu\nu}$ are expressed in terms of $F^{\mu\nu}$ from S using (13) then the same components are obtained as in (12).

5. Apparent transformations of electric and magnetic fields as bivectors

In contrast to the LT of $\mathbf{E}_H$ (and $\mathbf{B}_H$), Eqs. (10) and (11), it is accepted in the usual geometric algebra formalism that $\mathbf{E}_H$ (and $\mathbf{B}_H$) do not transform in the same way as all other multivectors, but that they transform as

$$\mathbf{E}'_{H,at} = (1/2)[F' - \gamma_0 F' \gamma_0] = (F' \cdot \gamma_0)\gamma_0, \tag{16}$$

where $F' = RF\widetilde{R}$. (The subscript "at" is for AT.) It is seen from (16) that only F is transformed while γ_0 is not transformed. The transformation (16) is nothing else than the usual transformation of the electric field that is given in Ref. [7], *Space-Time Algebra*, Eq. (18.22), *New Foundations for Classical Mechanics,* Ch. 9, Eqs. (3.51a,b) and Ref. [8], Sec. 7.1.2, Eq. (7.33).

When (16) is written with CBGQs, then instead of the LT (11) we find the AT

$$\mathbf{E}'_{H,at} = F'^{i0}\gamma_i \wedge \gamma_0 = E^1\gamma_1 \wedge \gamma_0$$

$$+\gamma(E^2 - \beta c B^3)\gamma_2 \wedge \gamma_0 + \gamma(E^3 + \beta c B^2)\gamma_3 \wedge \gamma_0, \tag{17}$$

In (17) we have used (4), i.e., that $F^{i0} = E^i$ and $(1/2c)\varepsilon^{kli0}F_{kl} = B^i$. When the transformed components $(\mathbf{E}'_{H,at})^{\mu\nu}$, $((\mathbf{E}'_{H,at})^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot \mathbf{E}'_{H,at}))$ from (17) are written in a matrix form they are

$$(\mathbf{E}'_{H,at})^{\mu\nu} = \begin{bmatrix} 0 & -E'^1_{at} & -E'^2_{at} & -E'^3_{at} \\ E'^1_{at} = F'^{10} & 0 & 0 & 0 \\ E'^2_{at} = F'^{20} & 0 & 0 & 0 \\ E'^3_{at} = F'^{30} & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

where

$$E'^1_{at} = E^1,\ E'^2_{at} = \gamma(E^2 - \beta c B^3),\ E'^3_{at} = \gamma(E^3 + \beta c B^2). \tag{19}$$

The same matrix form can be obtained for $(\mathbf{B}'_{H,at})^{\mu\nu}$ with

$$B'^1_{at} = B^1,\ B'^2_{at} = \gamma(B^2 + \beta E^3/c),\ B'^3_{at} = \gamma(B^3 - \beta E^2/c). \tag{20}$$

Observe that the transformations (19) and (20) are exactly the familiar expressions for the usual transformations of the components of the 3D $\mathbf{E}$ and $\mathbf{B}$, Ref. [4], Eq. (11.148), which are quoted in every textbook and paper on relativistic electrodynamics from the time of Lorentz [10], Poincaré [11, 12] and Einstein [9].

We see from (16), (17), (18), (19) and (20) that

(i′) $\mathbf{E}'_{H,at}$ and $\mathbf{B}'_{H,at}$, in the same way as $\mathbf{E}_H$ and $\mathbf{B}_H$ (see Eqs. (6)), are parallel to γ_0, i.e., $\mathbf{E}'_{H,at} \wedge \gamma_0 = \mathbf{B}'_{H,at} \wedge \gamma_0 = 0$, whence it again holds that the space-space components are zero, $(\mathbf{E}'_{H,at})^{ij} = (\mathbf{B}'_{H,at})^{ij} = 0$. Furthermore, it is seen from the relations (17), (19) and (20) that

(ii′) in contrast to the LT of $\mathbf{E}_H$ *and* $\mathbf{B}_H$, Eq. (11), *the components* E'^{i}_{at} *of the transformed* $\mathbf{E}'_{H,at}$ *are expressed by the mixture of components* E^i *and* B^i, *and the same holds for* $\mathbf{B}'_{H,at}$.

In all geometric algebra formalisms, e.g., [7, 8], the AT (17) for $\mathbf{E}'_{H,at}$ (and similarly for $\mathbf{B}'_{H,at}$) are considered to be the LT of $\mathbf{E}_H$ ($\mathbf{B}_H$). However, contrary to the generally accepted opinion, the transformations (16), (17), (18), (19) and (20) are not the LT. The LT cannot transform the matrix (5) with $(\mathbf{E}_H)^{ij} = 0$ to the matrix (18) with $(\mathbf{E}'_{H,at})^{ij} = 0$. Furthermore Eq. (14) is not fulfilled,

$$(1/2)(\mathbf{E}'_{H,at})^{\mu\nu}\gamma'_\mu \wedge \gamma'_\nu \neq (1/2)(\mathbf{E}_H)^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \tag{21}$$

which means that these two quantities are not connected by the LT, and consequently they do not refer to the same 4D quantity for relatively moving observers. As far as relativity is concerned, these quantities are not related to one another. The fact that they are measured by two observers (γ_0 - and γ'_0 - observers) does not mean that relativity has something to do with the problem. The reason is that observers in the γ_0-frame and in the γ'_0-frame are not looking at the same physical quantity but at two different quantities. *Every observer makes measurement on its own quantity and such measurements are not related by the LT.* The LT of $\mathbf{E}_H$ are correctly given by Eqs. (10), (11) and (12). Therefore, we call the transformations (16) and (17) for geometric quantities, and (19) and (20) for components, the "apparent" transformations, the AT.

The same name was introduced by Rohrlich [16] for the Lorentz contraction; the Lorentz contracted length and the rest length are not connected by the LT and consequently they do not refer to the same 4D quantity. Similar ideas about the Lorentz contraction were also raised by Gamba [17]. Rohrlich's and Gamba's ideas are generalized and properly formulated in geometric terms in Refs. [18, 19], where it is shown that not only the Lorentz contraction but the time dilatation is the AT as well. In Ref. [19], some of the well-known experiments: the "muon" experiment, the Michelson-Morley type experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments are analyzed using Einstein's formulation of special relativity with the Lorentz contraction and the time dilatation and the new one which exclusively deals with 4D AQs and 4D CBGQs. It is shown that, contrary to the general belief, all the experiments are in a complete agreement with the geometric formulation but not always with the usual formulation of special relativity.

In the usual covariant approaches [4], the components of the 3D $\mathbf{E}'$ and $\mathbf{B}'$ are identified in the same way as in (7), with six independent components of $F'^{\mu\nu}$, $E'_i = F'^{i0}$, $B'_i = (1/2c)\varepsilon_{ikl}F'_{lk}$. Such procedure then leads to the AT (19) and (20). The 3D $\mathbf{E}'$ and $\mathbf{B}'$, as *geometric quantities in the 3D space,* are constructed multiplying the components E'_i and B'_i by the unit 3D vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$. The important objections to such usual construction of $\mathbf{E}'$ and $\mathbf{B}'$ are the following: First, the components E'_i

and B_i' are determined by the AT (19) and (20) and not by the LT. Second, there is no transformation which transforms the unit 3D vectors **i**, **j**, **k** into the unit 3D vectors **i**′, **j**′, **k**′. Hence, it is not true that, e.g., the 3D vector $\mathbf{E}'=E_1'\mathbf{i}'+E_2'\mathbf{j}'+E_3'\mathbf{k}'$ is obtained by the LT from the 3D vector $\mathbf{E}=E_1\mathbf{i}+E_2\mathbf{j}+E_3\mathbf{k}$. Consequently, the 3D vectors $\mathbf{E}'$ and $\mathbf{E}$ are not the same quantity for relatively moving inertial observers, $\mathbf{E}'\neq\mathbf{E}$. Thus, although it is possible to identify the components of the 3D **E** and **B** with the components of F (according to Eq. (7)) in an arbitrarily chosen γ_0-frame with the $\{\gamma_\mu\}$ basis such an identification is meaningless for the Lorentz transformed F'.

6. *E and B as 1-vectors*

It is worth noting that the whole consideration is much clearer when using 1-vectors E and B, as in Refs. [2, 3], for the representation of the electric and magnetic fields. Then,

$$F = (1/c)E\wedge v + (IB)\cdot v,$$
$$E = (1/c)F\cdot v, \quad B = -(1/c^2)I(F\wedge v). \tag{22}$$

Here we shall only briefly consider the electric field. (Similar results hold for the magnetic field.) In the frame of "fiducial" observers, $E = F\cdot\gamma_0$, $E = E^i\gamma_i = F^{i0}\gamma_i$. By the active LT, the electric field E transforms again to the electric field (according to (9)) $E' = R(F\cdot\gamma_0)\widetilde{R} = F'\cdot\gamma_0'$, i.e., $E' = E'^{\mu}\gamma_\mu = -\beta\gamma E^1\gamma_0 + \gamma E^1\gamma_1 + E^2\gamma_2 + E^3\gamma$, which now contains the temporal component $E'^0 = -\beta\gamma E^1$. This is the way in which any 1-vector transforms. Generally, for components, $E'^0 = \gamma(E^0-\beta E^1)$, $E'^1 = \gamma(E^1-\beta E^0)$, $E'^{2,3} = E^{2,3}$. (In the usual covariant approach, these transformations are the LT of a 4-vector.) For the passive LT, it holds that $E = E^\mu\gamma_\mu = E'^\mu\gamma_\mu'$; E is the same quantity for relatively moving observers.

On the other hand, the AT (19) for components are obtained taking that $E_{at}' = F'\cdot\gamma_0$, only F is transformed but not γ_0, i.e., $E_{at}' = 0\gamma_0 + E_{at}'^i\gamma_i$, $E_{at}' = E^1\gamma_1 + \gamma(E^2-\beta cB^3)\gamma_2 + \gamma(E^3+\beta cB^2)\gamma_3$, and obviously E and E_{at}' are not the same quantity for relatively moving observers, $E^i\gamma_i\neq E_{at}'^i\gamma_i$. It is visible that all results with 1-vectors E and B are the same as those with bivectors E_v and B_v, but the procedure is much simpler and closer to the usual formulation with the 3D **E** and **B**. However, there is already extensive literature, e.g., Refs. [7, 8], in which the bivectors $\mathbf{E}_H$ and $\mathbf{B}_H$ are employed. Therefore, in this paper, the elaboration of the fundamental difference between the AT and the LT is mainly given using bivectors and not 1-vectors.

Additionally, the relations (1), (2) and (22) indicate that the electromagnetic field $F(x)$ (bivector) can be taken as the primary quantity for the whole electromagnetism from which the 4D geometric quantities, the electric and magnetic fields, are simply derived. Such a formulation was recently presented in Ref. [5], where a complete formulation of electromagnetism is developed from only one axiom, the field equation for the bivector field F.

7. *Conclusions*

The main conclusion that can be drawn from this paper, and Refs. [1 − 3], is that the usual transformations of the electric and magnetic fields are not the LT. It is believed in Refs. [7, 8], and many others, that the LT of the matrix of components $(\mathbf{E}_H)^{\mu\nu}$, Eq. (5), for which the space-space components $(\mathbf{E}_H)^{ij}$ are zero and $(\mathbf{E}_H)^{i0} = E^i$, transform that matrix to the matrix $(\mathbf{E}'_{H,at})^{\mu\nu}$, Eq. (18), in which again the space-space components $(\mathbf{E}'_{H,at})^{ij}$ are zero and the time-space components $(\mathbf{E}'_{H,at})^{i0} = E'^{\,i}_{at}$ are given by the usual transformations for the components of the 3D vector $\mathbf{E}$, Eq. (19); the transformed components $E'^{\,i}_{at}$ are expressed by the mixture of E^i and B^i components. (This statement is equivalent to saying that the transformations (19) and (20) are the LT of the components of the 3D $\mathbf{E}$ and $\mathbf{B}$.) However, according to the correct mathematical procedure, the LT of the matrix of components $(\mathbf{E}_H)^{\mu\nu}$, Eq. (5), transform that matrix to the matrix $(\mathbf{E}'_H)^{\mu\nu}$, Eq. (12), with $(\mathbf{E}'_H)^{ij} \neq 0$. As seen from (12), all transformed components $(\mathbf{E}'_H)^{\mu\nu}$ of the electric field are determined only by three components E^i of the electric field; there is no mixture with three components B^i of the magnetic field.

These results will be very surprising for all physicists since we are all, and always, taught that the transformations (19) and (20) are the LT of the components of the 3D $\mathbf{E}$ and $\mathbf{B}$. The whole physics community everyday deals with these AT considering that they are the correct relativistic transformations, i.e., the LT. But, the common belief is one thing and clear mathematical facts are a quite different thing. The true agreement of these new results with electrodynamic experiments, as shown in Refs. [2, 3] and Refs. [5, 6], substantially support the validity of the results from Refs. [1 − 3] and Refs. [5, 6]. It can be concluded that these results say that the Lorentz invariant 4D geometric quantities are physical ones, and not, as usually accepted, the 3D geometric quantities.

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LORENTZOVE I "PRIVIDNE" PRETVORBE ELEKTRIČNIH I MAGNETSKIH POLJA

Primjenom tenzorskog formalizma i Cliffordove, tj. geometrijske, algebre nedavno je utvrđeno da se uobičajene pretvorbe trodimenzijskih (3D) vektora električnih i magnetskih polja razlikuju od Lorentzovih pretvorbi odgovarajućih 4D veličina koje predstavljaju električna i magnetska polja. Primjenom formalizma geometrijske algebre, u ovom se radu istražuje ta osnovna razlika, predstavljajući električna i magnetska polja kao bivektore i 1-vektore.