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# GEOMETRY OF MANIFOLDS ON LIE ENDOMORPHISM SPACE AND THEIR DUALS UNDER FRACTIONAL ACTION-LIKE VARIATIONAL APPROACH

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Some interesting fractional features of the geometry of manifolds on Lie endomorphism space and their duals are discussed within the framework of fractional action-like variational approach (fractionally differentiated Lagrangian function) formulated recently by the author.

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## 1. *Introduction*

The purpose of this paper is to develop the fractional Euler-Poincaré systems on Lie algebras and their duals within the framework of fractionally differentiated Lagrangian function (or fractional action-like variational approach). The fractional mechanisms described through this work include some interesting features concerning weak dissipative or nonconservative dynamical systems and turbulent geophysical flows. The paper is organized as follows: after the Introduction, we review briefly the classical variational approach which fails to describe correctly weak dissipative and nonconservative dynamical systems; Sec. 3 is devoted to illustrate the importance of fractional calculus and its successful role in describing nonconservative systems; in Sec. 4, we introduce the reader to the fractional action-like variational approach concept where we review some of its important features and characteristics in particular the violation of Noether's symmetry theorems. The fractional Euler-Poincaré equations on Lie endomorphism space and their duals, which arise through a reduction of the fractional variational principle, are developed in Sec. 5, and some consequences are treated including the Kelvin-Noether's theorem, illustrated by a simple ideal plasma fluid flow example. In the last section, the fractional reduced Euler-Poincaré and fractional Hamel fractional equations on Lie algebras are developed. Finally, some discussion and outlook are presented in Sec. 7.

## 2. *The classical variational approach*

Historically, in the brachistochrone problem, the variational technique was used to find the path of quickest descent for a bead sliding along a wire under the action of gravity. The behavior of light and particles in a uniform medium (*geodesics, geometry, etc.*) was also discussed by Fermat within the concept of *Fermat's principle of least time*. After that comes the *Maupertuis-Jacobi's principle of least action* making use of the length variable $s$ as the orbit parameter to describe particle motion, and finally the celebrated Hamilton's principle from which equations of motion of any dynamical systems are derived in terms of generalized spatial coordinates [1−4]. The Lagrangian method was introduced in as a powerful alternative tool to the well-known Newtonian method for deriving equations of motion for complex mechanical systems. After that, a complementary approach to the Lagrangian method, known as the Hamiltonian method was introduced by Hamilton. In classical conservative Hamiltonian systems, the evolution of a classical physical system is usually described by a function $q(\tau)$, where "$q$" is a generalized coordinate and "$\tau$" is assumed here to be the intrinsic time. The trajectory $q(\tau)$ is in fact determined by assuming that the action functional $S[q(\tau)] = \int L(\tau, q(\tau), \dot{q}(\tau), \ddot{q}(\tau), ...)\,\mathrm{d}\tau, t_1 \leq \tau \leq t_2$ is extremized. $t_1$ and $t_2$ are two fixed moments of time such as $q(t_1) = q_1$ and $q(t_2) = q_2$ and the function $L$ is the Lagrangian of the dynamical systems. The requirement that the function $q(\tau)$ extremizes the action for $L(\tau, q(\tau), \dot{q}(\tau))$ leads to a differential equation for $q(\tau)$. If the function $q(\tau)$ is an extremum of the action functional, than a small perturbation, say $\delta q(\tau)$ will change the value of the action by terms which are quadratic in $\delta q(\tau)$. That is, the variation $\delta S[q, \delta q]$ should have no first-order terms in $\delta q(\tau)$. Thus with the boundary conditions $\delta q(t_1; t_2) = 0$ (*the variation $\delta q$ is assumed to vanish at the integration boundaries*) we obtain the celebrated Euler-Lagrange equations $\partial L/\partial q - \mathrm{d}/\mathrm{d}\tau\,(\partial L/\partial \dot{q}) = 0$. This is the classical equation of motion for any dynamical system having the Lagrangian $L(\tau, q(\tau), \dot{q}(\tau))$. In fact, the coordinates $q$ are based on a certain configuration manifold $Q$ and the Lagrangian is then the function $L : TQ \to R$ defined on the tangent bundle $TQ$ of $Q$ with canonical projection $\Gamma_Q : TQ \to Q$. Here $TQ$ is the velocity space with bundle coordinates $(q, \dot{q})$. The Hamiltonian formalism is based on the Legendre transform of the Lagrangian $L(\tau, q(\tau), \dot{q}(\tau))$ with respect to the velocity $\dot{q}(\tau)$. The time and position generalized coordinates do not participate in the Legendre transformation and they are kept as parameters. The momentum is defined as $p = \partial L/\partial \dot{q}$, that is $\dot{q} = v(p; q, \tau)$ ($v$ is the velocity), and the Hamiltonian function is defined by the following function $H(p, q, \tau) = \sum pv(p; q, \tau) - L(q, v(p; q, \tau), \tau)$. The resulting Hamilton equations of motion are $\dot{p} = -\partial H/\partial q$ and $\dot{q} = \partial H/\partial p$. An important consequence of classical conservative dynamical systems is Noether's theorem which states that for any symmetry of the Lagrangian, there corresponds a conservation law (*and vice versa*). In other words, when a system exhibits symmetry, then a constant of motion can be obtained. One of the most important consequences and well-known illustrations of this deep and important relation, is given by the conservation of energy in mechanics: the autonomous Lagrangian $L\left(\tau, q\left(\tau\right), \dot{q}\left(\tau\right)\right)$, correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry), and as a result $L(\tau, q(\tau), \dot{q}(\tau)) - \dot{q}\partial L(\tau, q(\tau), \dot{q}(\tau))/\partial\dot{q} \equiv \mathit{constant}$ follows directly from the Noether's theorem. This is to say that the total energy of a conservative system always remains constant in time. In other words, it cannot be created or destroyed, but only transferred from one form into another. That is, when the Lagrangian is invariant under a time translation, a space translation, or a spatial rotation, the conservation law involves energy, linear momentum, or angular momentum, respectively. One can mathematically prove that the Noether's conservation laws are still working if, for example, an extra desired term involving the nonconservative forces is added to the standard constants of motion. The last equation is in fact valid along all Euler-Lagrange extremals $q$ of an autonomous problem of the calculus of variations. The constant of motion is known in the standard calculus of variations as the second Erdmann necessary condition. It gains different interpretations in concrete applications: conservation of energy in classical and quantum mechanics, income-wealth law in economics, first law of thermodynamics and cosmology, etc. Nonconservative forces remove energy from the dynamical systems and, consequently, the constant of motion is broken. Unfortunately, for decaying, weak dissipative or nonconservatives dynamical systems, the above standard requirements and equations do not hold and didn't succeed to describe correctly the decaying behaviour. Other approaches are required.

## 3. *Importance of the fractional calculus*

Fractional calculus plays an important and leading role in the understanding of complex classical and quantum (conservative and nonconservative) dynamical systems with holonomic as well as with nonholonomic constraints [5 − 48]. Its origin goes back more than three centuries, when in 1695 L'Hospital asked Leibniz the mathematical meaning of $(\sqrt{\mathrm{d}/\mathrm{d}y})\,x$. After that, many famous mathematicians (J. Fourier, N. H. Abel, J. Liouville, B. Riemann, etc.) contributed strongly to the fractional analysis program, e.g. fractional derivatives and integrals. Although the fractional theory is very rich, it was considered for more than three centuries as a theoretical mathematical field with no physical interests. In the last few decades, fractional theory was proved to be very useful and important in various fields of science including classical and quantum physics, field theory, solid state physics, fluid dynamics, turbulence, chemistry in general, nonlinear biology, stochastic analysis, nonlinear control theory, image processing, etc. While various fields of application of fractional derivatives and integrals are already well developed, some others have just started, in particular the study of fractional problems of the calculus of variations (COV) and respective Euler-Lagrange type equations are a subject of current strong research and investigations. In 1996 − 97, F. Riewe used the COV with fractional derivatives and consequently obtained a version of the Euler-Lagrange equations (ELE) with fractional derivatives that combines the conservative and non-conservative cases [36]. In 2001 − 2002, another approach was developed by M. Klimek by considering fractional problems of the COV, but with symmetric fractional derivatives, and corresponding ELE's were obtained using both Lagrangian and Hamiltonian formalisms [38]. In 2002, O. Agrawal extended Klimek problem and proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [39]. In 2004 the ELE's of Agrawal were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [32]. In all above mentioned studies, ELE's depend on left and right fractional derivatives, even when the problem depend only on one type of them. In 2005, M. Klimek studied problems depending on symmetric derivatives for which ELE's include only the derivatives that appear in the formulation of the problem [33]. Fractional derivatives are used in all these approaches and, consequently, complicated mathematically the description of any dissipative or weak decaying dynamical systems. Remember that dissipative forces can be modelised in many different varieties of ways. Methodologically, dissipative Newtonian systems are nothing than complement to conservative systems, since not only energy, but also other physical quantities, such as linear and angular momentum, are not conserved. If, for instance, one considers an unconstrained Newtonian system with self-adjoint forces, then the Lagrangian exists if the fundamental analytic theorem for configurations space formulations is verified. A Lagrangian for its direct representation does not exist if the system is represented more realistically by adding, for example, a drag force linear in the velocity. In this way, the system tends to a non-self-adjoint situation. The problem of the existence of a typical Lagrangian depends in fact of whether it is self-adjoint or not and one refers to the transformation theory or the Lie algebra structure to study the required problem.

## 4. *Fractional action-like variational approach (FALVA)*

In a recent work, we developed a novel approach known as the fractional action-like variational approach (FALVA) or fractionally differentiated Lagrangian function (FDLF) to model and describe nonconservative Lagrangian dynamical systems within the framework of fractional differential calculus [49 − 59]. In our proposed method, fractional time integral introduces only one parameter "$\alpha$" while in other models an arbitrary number of fractional parameters (*orders of derivatives*) appears. The FALVA is based on the following concept:

**Definition 1**. Consider a smooth manifold $M$ (configuration space) and let $L : R \times TM \to R$ be the smooth Lagrangian function (smooth map). For any piecewise smooth path $q : [t_0, t_1] \to M$, we define the fractional action or fractionally differentiated Lagrangian function (FDLF) by

$$S_L\,[q] = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} L(\dot{q}(\tau), q(\tau), \tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau = \int_{t_0=0}^{t} L(\dot{q}(\tau), q(\tau), \tau) \mathrm{d}g_t(\tau) \,, \qquad (1)$$

where $L(\dot{q}, q, \tau)$ is the Lagrangian weighted with $(t-\tau)^{\alpha-1}/\Gamma_\alpha$ and $\Gamma(1+\alpha)g_t(\tau) = t^\alpha - (t-\tau)^\alpha$ with the scaling properties $g_{\mu t}(\mu\tau) = \mu^\alpha g_t(\tau)$, $\mu > 0$. In reality, we consider a smooth action integral (a time smeared measure $\mathrm{d}g_t(\tau)$ on the time interval $[0, t] \in R^+$) which can be rewritten as the strictly singular Riemann-Liouville type fractional derivative Lagrangian

$$S_{\beta\in(0,1)}[q] = D_t^{-1+\beta} L(\dot{q}(t), q(t), t)$$

$$= \int_0^t L\left(\dot{q}\left(t\right), q\left(t\right), t\right) \frac{\mathrm{d}\tau}{\left(t-\tau\right)^\beta} \overset{\beta\to 0}{\to} \int_0^t L\left(\dot{q}\left(t\right), q\left(t\right), t\right) \mathrm{d}\tau \,,$$

and thereby retrieve the standard action integral or functional integral. In this work, we have $\beta = 1-\alpha$, $\alpha \in (0,1)$. Such type of functionals is known in mathematical economy, describing, for instance, so called "discounting" economical dynamics. The true fractional derivatives are also often, nowadays, used for describing so called "dissipative structures" appearing in nonlinear dynamical systems, etc.

**Theorem 1**. *Let $L : R \times TM \to R$ be a Lagrangian and $x = x(\tau, \xi)$ be the coordinate point of the generalized $q(\tau, \xi)$. Set $L = L(x, y)$ evaluated at $y = \mathrm{d}x/\mathrm{d}\tau = \dot{x}$, where dot denotes time derivative with respect to $\tau$, $x = (x^i)$ is a coordinate system on $M$ and $y^i = \mathrm{d}x^i$ are functions on tangent vectors. Then the initial curves $x = x(\tau)$ satisfy the fractional Euler-Lagrange equations (FEL)*

$$E_i(L) \equiv \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial y^i}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial y^i} \equiv \frac{\partial R}{\partial y^i} \equiv F_{y^i,\alpha} \,, \qquad (2)$$

where $R(y^i) \equiv (1-\alpha)L(y^i)/(t-\tau)$ is identified as the Rayleigh dissipation function.

In fact, given $M$ and $L$, there is a unique function $E_L : TM \to R$ and a unique $1-form$ $\alpha_L$ on $TM$, such that in local coordinates, $(q_i, \dot{q}_i)$, we have $E(L) = \sum_i \dot{q}_i \partial L/\partial \dot{q}_i - L$ and $\alpha = (\partial L/\partial \dot{q}_i)\mathrm{d}q_i$. More generally, let $L(q^i, p_k^j)$ be a function on $C^\infty$ defined on $R^N \times R^{Np}$. What we are looking for is the extremum of the FDLF on a certain domain function $D$ with boundary $\partial D$ [4]

$$A_\alpha(\Lambda) = \int_{t_0=0}^{t} L(\Lambda^i(u(\tau)), \Lambda^j_{u^k}(u(\tau))) \mathrm{d}g_{u^1(\tau)} \wedge ... \wedge \mathrm{d}g_{u^p(\tau)} \,, \qquad (3)$$

where $\Lambda^j_{u^k} \equiv \partial\Lambda^j/\partial u^k$ and $\Lambda \in C^\infty : R^p \to R^N$ with generally $\mathrm{d}g_t(\tau) = (t-\tau)^{\alpha-1}\mathrm{d}\tau/\Gamma(\alpha)$. Consider now the elementary variations $\Lambda_s = \Lambda + \varepsilon\hat{\xi}$ of $\Lambda$ where

$\hat{\xi} : D \to R^3$ vanishes at the boundary $\partial D$ with $\varepsilon \ll 1$. The fractional directional derivative of $A_\alpha$ may be written as

$$DA_\alpha(\Lambda) \cdot \xi = \int\limits_{t_0=0}^{t} \left( \frac{\partial L}{\partial q^i} \xi^i + \frac{\partial L}{\partial p_k^j} \xi_{u^k}^j \right) \mathrm{d}g_{u^1(\tau)} \wedge ... \wedge \mathrm{d}g_{u^p(\tau)} \,. \tag{4}$$

Making use of the fractional Stokes formula $\int\limits_D \partial\omega = \int\limits_{\partial D} \omega$ with

$$\omega = \sum_{k=}^{p} (-1)^{k+1} \left( \frac{\partial L}{\partial \Lambda_{u^k}^j} \right) \mathrm{d}g_{u^1(\tau)} \wedge ... \wedge \mathrm{d}g_{u^p(\tau)} \,, \tag{5}$$

one finds easily

$$DA_\alpha(\Lambda) \cdot \xi = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0=0}^{t} \left\{ \left( \frac{\partial L}{\partial q^i} - \frac{\partial}{\partial u^k} \left( \frac{\partial L}{\partial p_k^i} \right) - \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial p_k^i} \right) \right\} \hat{\xi}^i \mathrm{d}u^1(\tau) \wedge ... \wedge \mathrm{d}u^p(\tau) \tag{6}$$

The critical points are then solutions of the fractional Euler-Lagrange equations (2) with $1 \leq i \leq N$.

**Lemma 1**. *If $\left(q^i, \dot{q}^i\right)$ are holonomic local coordinates on TM such that $\gamma(\tau) = \left(q^i(\tau)\right)$ and $\dot{\gamma}(\tau) = \left(\dot{q}^i(\tau)\right)$, then $\gamma$ is a solution of the fractional systems of nonlinear ordinary differential equations in one of the following forms:*

$$\frac{\partial L}{\partial q^k} \left(\dot{q}^j; q^j, \tau\right) - \frac{\mathrm{d}}{\mathrm{d}\tau} \left( \frac{\partial L}{\partial \dot{q}^k} \right) \left(\dot{q}^j; q^j, \tau\right) = \frac{1-\alpha}{t-\tau} \frac{\partial L}{\partial \dot{q}^k} \left(\dot{q}^j; q^j, \tau\right) \,, \tag{7}$$

$$\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^l} \ddot{q}^l \left(\dot{q}^j; q^j, \tau\right) + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^l} \dot{q}^l \left(\dot{q}^j; q^j, \tau\right) - \frac{\partial L}{\partial q^k} \left(\dot{q}^j; q^j, \tau\right) + \frac{\alpha-1}{\tau-t} \frac{\partial L}{\partial \dot{q}^k} \left(\dot{q}^j; q^j, \tau\right) = 0 \,, \tag{8}$$

where $k = 1, ..., n = \dim M$. *Dot denotes time derivative with respect to $\tau$.*

**Definition 2**: *A path $q : [t_0, t_1] \to M$ satisfying Eq. (2) is said to be a fractional extremal of the Lagrangian L.*

**Definition 3**: *The fractionally differentiated Lagrangian function (1) is said to be quasi-invariant under the infinitesimal $\tilde{\varepsilon}$-parameter transformations* [59] :

$$\bar{\tau} = \tau + \tilde{\varepsilon}\kappa(\tau, q) + O\left(\tilde{\varepsilon}^2\right) \,, \tag{9}$$

$$\bar{q}(\bar{\tau}) = q(\tau) + \tilde{\varepsilon}\omega(\tau, q) + O\left(\tilde{\varepsilon}^2\right) \,, \tag{10}$$

*up to a gauge term $\Lambda$ if, and only if*

$$L\left(\bar{\tau}, \bar{q}^k(\bar{\tau}), \bar{q}'^k(\bar{\tau})\right) (t-\bar{\tau})^{\alpha-1} \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\tau}$$

$$= L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)(t-\tau)^{\alpha-1} + \tilde{\varepsilon}(t-\tau)^{\alpha-1}\frac{\mathrm{d}\Lambda}{\mathrm{d}\tau}\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right) + O\left(\tilde{\varepsilon}^2\right).$$

**Theorem 2**: *If the FDLF represented by equation (1) is invariant up to a gauge term $\Lambda$, and if*

$$L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right) = -\left(\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)/\dot{q}\right)\cdot(\omega - \dot{q}\kappa),$$

*then*

$$\frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k}\cdot\omega\left(\tau, q^k\right)$$

$$+\left[L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right) - \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k}\cdot\dot{q}^k\right]\kappa\left(\tau, q^k\right) - \Lambda\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)$$

*is a constant of motion.*

In fact, the conservation of momentum, when the Lagrangian is not a function of the generalized coordinates $q^k$, or conservation of energy when the Lagrangian has no explicit dependence on time $\tau$, are no more true for a fractional order of integration $\alpha \neq 1$. When the Lagrangian is not a function of $q^k$, the constant of motion takes the form

$$\frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} + (1-\alpha)\int \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k}\frac{1}{t-\tau}\mathrm{d}\tau.$$

For $\alpha = 1$, $\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)/\partial \dot{q}^k$ is constant and $L$ is conserved.

**Theorem 3**: *Let $\gamma(\tau)$ be an extremal of a time-independent Lagrangian $L = L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)$. Then, if the Hamiltonian $H\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right) = p_k\dot{q}^k - L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)$ of the system does not have any explicit dependence on time, and if we drop terms higher than $\dot{q}^k$, then the Hamiltonian is not conserved $(H : T^*M \to R)$.*

***Proof:*** *Let $T \equiv t - \tau$ be a time-change of the variable. Using the facts*

$$\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)/\partial\tau = 0$$

*and*

$$\frac{\mathrm{d}L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\mathrm{d}\tau} = \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial\tau} + \sum\nolimits_k \dot{q}^k\left(\frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial q^k}\right),$$

*we obtain:*

$$\frac{\mathrm{d}H\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\mathrm{d}T} \equiv \frac{\mathrm{d}}{\mathrm{d}T}\left\{\sum_k \dot{q}^k\frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} - L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)\right\}$$

$$= \sum_k \dot{q}^k \left(\frac{1-\alpha}{T}\right) \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} = \frac{1-\alpha}{T} \sum_k p_k \dot{q}_k \, . \quad \blacksquare$$

*When $\alpha = 1$, the energy is conserved.*

**Corollary 1**: *The equations of motion are reduced to the pair of fractional canonical Hamilton's equations:*

$$\dot{p}^k = -\frac{\partial H}{\partial q^k} - \frac{1-\alpha}{t-\tau} p^k \, , \tag{11}$$

$$\dot{q}^k = \frac{\partial H}{\partial p^k} \, . \tag{12}$$

**Definition 4**: *The fractional Newton's second law is defined as*

$$m\ddot{q}^k = -\frac{\partial U}{\partial q^k} + A(T) m \dot{q}^k \, , \tag{13}$$

*where for simplicity $A(T) \equiv (1-\alpha)/T$, and $U$ represents the classical potential.*

Eq. (13) can be written also in the following form

$$\ddot{q}^k + \frac{\alpha - 1}{T} \dot{q}^k \equiv \frac{1}{T^{\alpha-1}} \frac{\mathrm{d}}{\mathrm{d}T} \left(T^{\alpha-1} \dot{q}^k\right) = -\frac{1}{m} \frac{\partial U}{\partial q^k}. \tag{14}$$

In other words, consider the Lagrangian $L = 1/2\, |\dot{q}|^2 - U(q)$, $U(q)$ being the potential energy. The fractional ENEL equations become $m\left(\ddot{q} + ((\alpha-1)/(\tau - t))\, \dot{q}\right) = -\nabla U(q)$ which are the modified Newton's equation of motion of a particle of mass $m$ in the force field $-\nabla U$ generated by the potential $U$.

**Corollary 2**: *For a Hamiltonian system with $H = H\left(p^k, q^k\right)$ on a typical $2n$-dimensional symplectic manifold $(M, \Xi)$ with the symplectic structure $\Xi$, the trajectory is determined by the fractional action principle provided that Eqs. (11) and (12) are satisfied. For arbitrary curves on $M$, the "$\alpha$ –Euler-Lagrange $1-form$"*

$$E = \left(\dot{q}^k - \frac{\partial H}{\partial p_k}\right) \mathrm{d}p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q^k} + \frac{1-\alpha}{t-\tau} p_k\right) \mathrm{d}q^k = \dot{q}^i \mathrm{d}p_i - \dot{\hat{p}}_i - \mathrm{d}H \, , \tag{15}$$

with

$$\dot{\hat{p}}_1 = \dot{p}_i + \frac{1-\alpha}{t-\tau} p_i \equiv \dot{p}_i + \frac{\alpha-1}{T} p_i = \frac{1}{T^{\alpha-1}} \frac{\mathrm{d}}{\mathrm{dT}} \left(T^{\alpha-1} p_i\right) , \tag{16}$$

*can then be defined along a curve, and $E = 0$ gives rise the canonical Eqs. (11) and (12).*

**Theorem 4**: *Let $L$ be a Lagrangian on $TM$ and $X$ a vector field on $M$. If the Lagrangian $L$ is $X$-invariant, then the quantity $P_X = X^k p_k$ is not conserved along the extremals of $L$.*

**Proof**: *Let $\gamma = \gamma\left(q^k(\tau)\right)$ be an extremal of the Lagrangian $L$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} P_X(\gamma, \dot{\gamma}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left( X^k\left(\gamma(\tau)\right) \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} \right)$$

$$= \frac{\partial X^k}{\partial q^k} \dot{q}^k \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} + X^k \left( \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial q^k} - \frac{\alpha - 1}{\tau - t} \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} \right)$$

$$= \frac{1-\alpha}{\tau - t} X^k \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k},$$

*assuming $X$-invariant, e.g.*

$$\frac{\partial X^k}{\partial q^k} \dot{q}^k \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial \dot{q}^k} + X^k \frac{\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)}{\partial q^k} = 0\,. \;\blacksquare$$

**Corollary 3**: *Let $L = L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)$ be a Lagrangian on the set $R^n$. If $\partial L\left(\tau, q^k(\tau), \dot{q}^k(\tau)\right)/\partial q^k = 0$, then $\mathrm{d}p^k/\mathrm{d}\tau \neq 0$ along any fractional extremal.*

## 5. *The fractional Euler-Poincaré on Lie endomorphism space and their dual*

Let $G_{\mathrm{Lie}}$ be a Lie group (a smooth manifold $G$), $g$ its Lie algebra and $g^*_{\mathrm{Lie}}$ its dual. The corresponding smooth group structure is described by the map

$$\begin{array}{ll} G \times G \to G\,, & G \to G\,, \\ (g,h) \to gh\,, & g \to g^{-1}\,. \end{array}$$

We summarize the fractional formalism with the following theorem [60–62].

**Theorem 5**: *The following conditions are equivalent*:

**A-The Fractional Hamilton's Principle**:
*Any curve $r(\tau) \in G_{\mathrm{Lie}}$ is a critical point of the fractional action*

$${}_1S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0=0}^{t} L\left(\dot{r}(\tau), r(\tau), \tau\right)(t - tau)^{\alpha-1} \mathrm{d}\tau,$$

*for variations $\delta r(\tau)$ such that $\delta r(\tau_0) = \delta r(\tau_1) = 0$.*

**B-The Fractional Euler-Poincaré variational principle**:

*Any curve* $r(\tau) \in G_{\rm Lie}$ *is a critical point of the fractional action*

$$ {}_2S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0=0}^{t_1} l(\mathrm{v}(\tau),\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau\,, $$

*for variations* $\delta r(\tau)$ *such that* $\delta r(\tau_0) = \delta r(\tau_1) = 0$.

**C-The Factional Hamilton's phase space principle**:

*The curve* $(r(\tau), p(\tau)) \in T^*G$ *is a critical point of the fractional action*

$$ {}_3S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0=0}^{t_1} \left(p\cdot\dot{r} - H(r,p)\right)(t-\tau)^{\alpha-1}\mathrm{d}\tau\,, $$

*for variations* $(\delta r, \delta p)$ *such that* $\delta r\,(\tau_i) = 0, i = 0,1$ *and* $\delta p(\tau)$ *is arbitrary. The pointwise function is defined on* $TG_{\rm Lie} \oplus T^*G_{\rm Lie}$ *regarded as a bundle over* $G_{\rm Lie}$*, that is, the base space common to* $TG_{\rm Lie}$ *and* $T^*G_{\rm Lie}$.

**D-The Fractional Lie-Poisson Variational Principle**:

*The curve* $(\mathrm{v}(\tau), \varsigma(\tau)) \in g_{\rm Lie} \times g^*_{\rm Lie}$ *is a critical point of the fractional action*

$$ {}_4S_{0<\alpha<1}[q] = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0=0}^{t_1} \left(\langle\varsigma(\tau),\mathrm{v}(\tau)\rangle - h\left(\varsigma(\tau)\right)\right)(t-\tau)^{\alpha-1}\mathrm{d}\tau\,, $$

*for variations* $\delta\mathrm{v}(\tau) = \dot{\kappa}t(\tau) + [\mathrm{v}(\tau),\kappa(\tau)]$ *such that* $\kappa\,(\tau_i) = 0, i = 0,1$ *and* $\delta p(\tau)$ *are arbitrary.*

*The bracket is the familiar Poisson one and for any Hamiltonian*

$$ h : g^*_{\rm Lie} \to R\,\langle\delta h/\delta\mu,\nu\rangle = dh(\mu)\cdot\nu, $$

*where* $dh(\mu) : g^*_{\rm Lie} \to R$ *is the usual derivative of* $h$*, and* $\langle\ ,\rangle$ *denotes the pairing between* $g_{\rm Lie}$ *and* $g^*_{\rm Lie}$.

**E-The Fractional Euler-Lagrange Equations on** $G_{\rm Lie}$ :

$$ \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial\dot{r}}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial\dot{r}} $$

**F-The Fractional Euler-Poincaré Equations on** $g^*_{\rm Lie}$ :

$$ \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial\mathrm{v}}\right) + \frac{\alpha-1}{\tau-t}\frac{\partial L}{\partial\mathrm{v}} = \mathrm{ad}^*_{\mathrm{v}}\frac{\partial \mathrm{L}}{\partial\mathrm{v}}\,, $$

*where* $\mathrm{v}(\tau) = \mathrm{r}^{-1}(\tau)\dot{\mathrm{r}}(\tau)$. *Here for a given* $\xi$*, the adjoint action of* $G_{\rm Lie}$ *on* $g_{\rm Lie}$ *is defined by*

$$ \mathrm{ad_g}\,(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} g\exp\left(\xi\tau\right)g^{-1}, $$

*so that*

$$ \mathrm{ad_g} : g_{\rm Lie} \to Endomorphism\,(g) $$

*is an invertible linear map and defines a left group action on* $g_{\rm Lie}$. *This holds also for the dual* $g^*_{\rm Lie}$.

**G-The Fractional Hamilton's Equations on $T^*G_{\rm Lie}$:**

$$\left(\frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau}, \frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau}\right) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial r} - \frac{1-\alpha}{t-\tau} p\right)$$

**H-The Fractional Lie-Poisson on $g^*_{\rm Lie}$:**

$$\dot{\mu} + \frac{\alpha-1}{\tau-t}\mu = \mathrm{ad}_{\delta h}/\delta\mu^* \mu$$

In fact, for $\xi \in g_{\rm Lie}$, the fractional Euler-Poincaré equations on $g^*_{\rm Lie}$with basis $e_1, ..., e_p (\dim g_{\rm Lie} = r)$ may be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \xi}\right) = \mathrm{ad}^*_\xi \frac{\partial \mathrm{L}}{\partial \xi} + \mathrm{f}\,, \tag{17}$$

where $f \equiv ((1-\alpha)/(\tau - t))\,\partial L/\partial\xi$. In the finite-dimensional case, Eq. (17) reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \xi^d}\right) = C^b_{ad}\frac{\partial L}{\partial \xi^b}\xi^a + f_d\,, \tag{18}$$

where $C^b_{ad}$ are the dual constants of the Lie algebra defined by $[e_a, e_b] = C^d_{ab} e_d$, $a, b = 1, ..., p$ and a sum over $d$ is understood. For example, for a Lie algebra $R^3$ with the corresponding rotational group and $L : R^3 \to R$, Eqs. (18) generalize the fractional Euler-equations for rigid body with quadratic Lagrangians or the usual Euler-equations with decaying external forces. In fact, the fractional equations describing the Euler rigid body are equivalent to the rigid body fractional action-like principle (1). Let $V$ be the vector space and assume that $g_{\rm Lie}$ acts on the left by linear maps on $V$ and on its dual space $V^*$. Define the linear map $\rho_v = g \to V$ defined by $\rho_v(\xi) = \xi v$ and its dual $\rho^*_v = g^* \to V^*$ [63, 64]. For any $a \in V^*$, making use for convenience of the celebrated bilinear notational operation $\rho^*_v a = v\diamondsuit a \in g^*_{\rm Lie}$, the positive and negative fractional Lie-Poisson on $g^*_{\rm Lie}$ may be written, respectively as [65, 66]

$$\dot{\mu} + \frac{\alpha-1}{\tau-t}\mu = \mp \mathrm{ad}_{\delta h/\delta\mu}\diamondsuit a\,, \tag{19}$$

$$\dot{a} = \pm\frac{\delta h}{\delta\mu} a\,. \tag{20}$$

The fractional Euler-Poincaré equations hold also on $g_{\rm Lie} \times V^*$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \xi}\right) = \mathrm{ad}^*_\xi \frac{\partial L}{\partial \xi} + \frac{\delta L}{\delta a}\diamondsuit a + f\,, \tag{21}$$

with $\dot{a} = -\xi(\tau) a$. In a continuum medium, Eq. (21) is written as

$$\frac{\partial}{\partial\tau}\left(\frac{\delta L}{\delta \vec{u}}\right) = \mathrm{ad}^*_{\vec{u}}\frac{\delta L}{\delta\vec{u}} + \frac{\delta L}{\delta a}\diamondsuit a + f = -\pounds_{\vec{u}}\frac{\delta L}{\delta\vec{u}} + \frac{\delta L}{\delta a}\diamondsuit a + f_{\vec{u}}\,, \tag{22}$$

where $\pounds_{\vec{u}}$ denotes the Lie derivative with respect to the Eulerian velocity field $\vec{u}(x,\tau)$ and $f_{\vec{u}} = f \equiv ((1-\alpha)/(\tau - t))\, \delta L/\partial \vec{u}$. Equations (17)–(22) may describe weak dissipative plasma fluid dynamics. One interesting consequence of the above arguments is the violation of the Kelvin-Noether theorem (conservation of circulation). It is worth mentioning that in continuum theories, the Kelvin circulation theorem for ideal current flow is related to the Noether theorem applied to continuum medium using the particle relabelling group of symmetry [62, 63, 67–76]. In fact, the Kelvin-Noether theorem has many important consequences in geophysical fluid dynamics and turbulent fluids with dissipation produced by vortex line-stretching. However, existing proofs of Kelvin-Noether conservation theorem are valid only for ideal (smooth and laminar) fluids. In fact, recent numerical simulations done for high-Reynolds-number turbulence show that the classical Kelvin-Noether theorem is violated [77]. Define the Kelvin-Noether quantity $I : C \times g \times V^* \to R$ by [60]

$$I\,(c,\xi,a) = \left\langle K(c,a), \frac{\partial L}{\partial \xi}\,(\xi,a) \right\rangle. \tag{23}$$

$C$ is a manifold on which the Lie group $G$ acts on the right, $K : C \times V^* \to g^{**}$ is an equivariant map and $g^{**}$ is the double dual Lie algebra.

**Corollary 4**: *If $\xi(\tau)$, $a(\tau)$ satisfy the fractional Euler-Poincaré equations, then the Kelvin quantity $I(\tau) = I\,(c(\tau),\xi(\tau),a(\tau))$ with $I : C \times g \to R$ is not conserved. Here $c(\tau) = c_0 g(\tau)^{-1}, c_0 \in C$ and $\dot{g}(\tau) = \xi(\tau) g(\tau)$. Conservation occurs for $\alpha = 1$ (standard case) or when $\tau \to \infty$.*

**Theorem 6**: *If the Eulerian velocity field $\vec{u}(x,\tau)$ satisfies the fractional Euler-Poincaré equations $\frac{\partial}{\partial \tau}\left(\frac{\delta L}{\delta \vec{u}}\right) = -\pounds_{\vec{u}} \frac{\delta L}{\delta \vec{u}} + \frac{\delta L}{\delta a} \diamondsuit a + f_{\vec{u}}$, and the fractional continuity equation*

$$\frac{\partial a}{\partial \tau} + \pounds_{\vec{u}}\, a = 0\,,$$

*then*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(I(\tau) \equiv I\left(\Im_\tau, \vec{u}_\tau, a_\tau\right)\right) = \oint\limits_{\Im_\tau} \frac{1}{\rho}\left(\frac{\delta L}{\delta a} \diamondsuit a + f_{\vec{u}}\right),$$

*where $\Im_\tau = \aleph_\tau \circ \Im_0$, $\aleph_\tau$ is the flow of $\vec{u}(x,\tau)$ and $\rho$ is the mass density.*

**Proof:** *Following [64], we write the circulation and we perform variable change as*

$$I\left(\Im_\tau, \vec{u}_\tau, a_\tau\right) = \oint\limits_{\Im_\tau} \frac{1}{\rho_\tau} \frac{\delta L}{\delta \vec{u}} = \oint\limits_{\Im_0} \frac{1}{\rho_0} \aleph_\tau^* \frac{\delta L}{\delta \vec{u}}$$

*Then making use of the definition of the Lie derivative*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)_{\mathrm{Lie}} (\aleph_\tau^* \zeta_\tau) = \aleph_\tau^* \left(\frac{\partial}{\partial \tau}\zeta_\tau + \pounds_{\vec{u}} \zeta_\tau\right),$$

*we easily find using the fractional Euler-Poincaré equations and the change of variables*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} I\left(\Im_\tau, \vec{u}_\tau, a_\tau\right) = \frac{\mathrm{d}}{\mathrm{d}\tau} \oint\limits_{\Im_0} \frac{1}{\rho_0} \aleph_\tau^* \frac{\delta L}{\delta \vec{u}} = \oint\limits_{\Im_0} \frac{1}{\rho_0} \frac{\mathrm{d}}{\mathrm{d}\tau} \left( \aleph_\tau^* \frac{\delta L}{\delta \vec{u}} \right)$$

$$= \oint\limits_{\Im_0} \frac{1}{\rho_0} \frac{\mathrm{d}}{\mathrm{d}\tau} \left( \frac{\partial}{\partial \tau} \frac{\delta L}{\delta \vec{u}} + \pounds_{\vec{u}} \frac{\delta L}{\delta \vec{u}} \right) = \oint\limits_{\Im_\tau} \frac{1}{\rho} \left( \frac{\delta L}{\delta a} \diamondsuit a + f_{\vec{u}} \right). \blacksquare$$

The circulation is not conserved as it is expected. One may even prove that the potential vorticity on fluid parcels is violated. The above fractional formalism and its nonconservation counterparts may be useful to describe dissipative phenomena, like those occurring in flows in porous media and variable density multi-species flows [66, 74, 76, 78]. An important simple implication concerns the ideal plasma fluid with pressure $p$. Given the reduced Lagrangian $l(u) = (1/2) \int\limits_B \left\| u^2 \right\| \mathrm{d}\mu$ with $g_{\mathrm{Lie}} = diffeomorphisme_{vol}(B)$, $B$ being the Riemann manifold. In the presence of a force term $F$, Eq. (17) is equivalent to [66]

$$\frac{\partial u}{\partial \tau} + u \cdot \nabla u = f + F - \nabla p\ . \tag{24}$$

Multiplying this equation by the fluid density, the resulting differential equation will express the violation of the conservation of momentum in a fluid flow. For a fluid at rest ($u = 0$), the balance equation between the volume and pressure forces, describing the stress condition in fluid at rest, is no longer valid, i.e. $f$ decreases with time. One may state the problem differently: the force $F$ may be written as

$$F = F_0 + \frac{1-\alpha}{\tau - t} p\,, \tag{25}$$

where $p$ is the momentum and $F_0$ is the value of $F$ when $\alpha = 1$, i.e. $F \to F_0$ for $\alpha = 1$ (standard action) or when $\tau \to \infty$. The second term on the RHS of Eq. (25) may be viewed as a perturbed force term in the theory. The above fractional formalism could be useful to explain the breakdown of magnetic flux conservation for ideal magnetohydrodynamics (MHD) plasmas and energy dissipation anomaly in hydrodynamic turbulence [79]. Other applications may be discussed in fluid and plasma dynamics. We leave this for a future work.

## 6. *The Fractional reduced Euler-Poincaré and Hamel equations on Lie algebras*

One can describe the above concepts of fractional Hamiltonian's mechanics in the context of Lie algebras and their duals, that is by writing the equations directly on the Lie algebra, by passing the Lie-Poisson equations on the dual. The derivations will be based on the reduction of fractional variational principles. Let $L : TSO(3) \to R$ be the Lagrangian of a rigid body motion written entirely in terms of the body angular velocity $\Omega$. We assume that the curve $\gamma(\tau) \in SO(3)$ satisfies the ENEL equations. If we denote by $M_s = \gamma \; \overline{\overline{I}} \; \Omega$ the spatial angular momentum, where $\overline{\overline{I}}$ is the symmetric moment of inertia tensor, then ENLE yields a non-conservation of $M_s$, e.g. $\mathrm{d}M_s/\mathrm{d}\tau \neq 0$ [60−64].

**Theorem 7**. *The Euler's equation* $\overline{\overline{I}} \; \dot{\Omega} = \overline{\overline{I}} \; \Omega \times \Omega$ *is equivalent to conservation of the spatial angular momentum unless* $\alpha = 1$. *Otherwise, the equivalence does not hold.*

**Theorem 8**: *If the curve* $\gamma(\tau) \in SO(3)$ *satisfies the FEL with*

$$L(\gamma, \dot{\gamma}) = \frac{1}{2} \int\limits_B \rho(X) \left\| \dot{\gamma} X \right\|^2 \mathrm{d}^3 X \,, \tag{26}$$

*if and only if* $\Omega(\tau)$ *defined by* $\gamma^{-1}\dot{\gamma}u = \Omega \times u, \; \forall \; u \in R^3$ *satisfies the fractional Euler's equation* $\overline{\overline{I}} \; \mathrm{d}\dot{\hat{\Omega}}/\mathrm{d}\tau = \overline{\overline{I}} \; \Omega \times \Omega$, *where* $\dot{\hat{\Omega}} = T^{1-\alpha}\mathrm{d}/\mathrm{d}T(T^{\alpha-1}\Omega)$. $\rho(X)$ *is a given mass density and* $B \subset R^3$ *is a reference configuration.*

**Proof:** *This is evident due to the fact that Noether's theorem for the spatial angular momentum is not conserved and from this one derives* $\overline{\overline{I}} \; \mathrm{d}\dot{\hat{\Omega}}/\mathrm{d}\tau = \overline{\overline{I}} \; \Omega \times \Omega$ ■.

**Theorem 9**: *Let* $G$ *be a Lie group,* $g = T_e G$ *its corresponding Lie algebra,* $L : TG \to R$*a left invariant Lagrangian,* $l : g \to R$ *be its restriction to the identity,* $\gamma(\tau) \in G$ *and* $\xi(\tau) = \gamma(\tau)^{-1} \cdot \dot{\gamma}(\tau)$. *Then, if* $\gamma(\tau)$ *satisfies the FEL, the corresponding Euler-Poincaré equations do not hold*

$$\mathrm{ad}_{\xi}^{*} \frac{\partial L}{\partial \xi} \neq \frac{\mathrm{d}}{\mathrm{d}\tau} \left( \frac{\partial L}{\partial \xi} \right) . \tag{27}$$

*where* $\mathrm{ad}_{\xi} : g \to g = \mathrm{ad}_{\xi} A = [\xi, A]\,; [,] : g \times g \to g$ *is the associated Lie bracket, and* $\mathrm{ad}_{\xi^*}$ *is its dual.*

**Corollary 5**: *The fractional variational principle for* $l(\xi(\tau))$ *holds on* $g$ *and on the tangent bundle of any configuration manifold* $Q$, *using* $\delta\xi = \dot{\eta} + ((\alpha - 1)/T)\,\eta + [\xi, \eta]$, *where* $\eta$ *vanishes at the endpoints.*

**Corollary 6**: *The spatial angular momentum is not conserved. We have*

$$\mathrm{d}\bar{M}_s/dT \propto T^{1-\alpha}, \qquad \bar{M}_s = \mathrm{Ad}^{*}_{g^{-1}} \partial l/\partial \xi.$$

This is an anomalous spatial angular momentum and it does have the property that, if $\bar{M}_s$ does not depend on $q_i$, it is not a constant of motion. Since the FEL and fractional Hamilton's equations are equivalent on the tangent bundle configuration space, then the fractional Lie-Poisson and fractional Euler-Poincaré equations are also equivalent on $TQ$ and $T^*Q$. In reality, the standard Lie-Poisson and Euler-Poincaré equations ($\alpha = 1$), as well as their combination, occur for many interesting conservative physical systems and had revealed interesting features from the

Lagrangian point of view. The basic idea is to drop Euler-Lagrange equations and variational principles from a general velocity phase space $TQ$ to the quotient $TQ/G$ by a Lie group action of $G$ on $Q$. If the Lagrangian is a $G$-invariant Lagrangian on the tangent bundle $TQ$, it induces a particular reduced Lagrangian on $TQ/G$ by implementing a specific connection on the principal bundle $Q \to S = Q/G$, assuming that this quotient is nonsingular, allowing one to split the variables into a horizontal and vertical part. Note that this formalism holds only for conservative systems and it fails for dissipative or nonconservative dynamical systems. Let $\left(x^\beta, \eta^a\right)$ be internal coordinates for the shape space $Q/G$ and for the Lie algebra $g$ relative to a chosen basis, respectively, $l = l\left(x^\alpha, \dot{x}^\alpha, \eta^a\right)$ the Lagrangian and $C^a_{db}$ the structure constants of the Lie algebra.

**Definition 5**: *The resulting fractional Hamel equations are*

$$\frac{\partial l}{\partial x^\beta} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial l}{\partial \dot{x}^\beta}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial l}{\partial \dot{x}^\beta}\,, \tag{28}$$

$$\frac{\partial l}{\partial \eta^a} C^a_{db}\eta^d - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial l}{\partial \eta^b}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial l}{\partial \eta^b}\,, \tag{29}$$

*and the fractional reduced FEL equations are*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial l}{\partial \dot{x}^\beta}\right) - \frac{\partial l}{\partial x^\beta} \equiv \frac{\partial l}{\partial \xi^a}\left(B^a_{\beta\delta}\dot{x}^\delta + B^a_{\beta d}\xi^d\right) = \frac{\alpha-1}{t-\tau}\frac{\partial l}{\partial \dot{x}^\beta}\,, \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial l}{\partial \xi^b}\right) \equiv \frac{\partial l}{\partial \xi^a}\left(B^a_{\beta b}\dot{x}^\beta + C^a_{db}\xi^d\right) = \frac{\alpha-1}{t-\tau}\frac{\partial l}{\partial \xi^b}\,, \tag{31}$$

*which are equivalent respectively to:*

$$\frac{1}{T^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}T}\left(T^{\alpha-1}\frac{\partial l}{\partial \dot{x}^\beta}\right) - \frac{\partial l}{\partial x^\beta} \equiv \frac{\partial l}{\partial \xi^a}\left(B^a_{\beta\delta}\dot{x}^\delta + B^a_{\beta d}\xi^d\right), \tag{32}$$

$$\frac{1}{T^{\alpha-1}}\frac{\mathrm{d}}{\mathrm{d}T}\left(T^{\alpha-1}\frac{\partial l}{\partial \xi^b}\right) \equiv \frac{\partial l}{\partial \xi^a}\left(B^a_{\beta b}\dot{x}^\beta + C^a_{db}\xi^d\right), \tag{33}$$

*where $\xi^a = A^a_\beta \dot{x}^\beta + \eta^a$ is the velocity shift, $A^d_\beta$ are the local coordinates of the connection $A$ and $B^a_{\beta d} = C^a_{bd}A^b_\beta = -B^a_{d\beta}$ are the coordinates of the curvature $B$ of $A$.*

In this way, the reduction of a cotangent bundle $T^*Q$ by a symmetry group $G$ is also a bundle over $T^*S : S = Q/G$ unless Eqs. (30)−(33) are satisfied, i.e. the FDLF holds for any weak dissipative systems. The applications described here are simple but serve as a staring point.

## 7. *Conclusions and perspectives*

The nonconservative equations constructed through this work have the essential feature that both energy and angular momentum conservation laws are violated. In the context of Euler-Poincaré, this means that the energy is decreasing along the coadjoints non-invariant orbits. For example, in plasma physics and stellar dynamics, we enlarged our arguments by stating that dissipative mechanism occurs with simultaneous violation of particle number and energy [79 − 82]. In geophysical scenarios, both energy and entropy are violated [81]. We expect important implications of our formalism to a number of known examples, including the heavy top [67 − 73], compressible and incompressible fluids, inviscid, irrotational, unsteady flows, plasma MHD, superfluid turbulence [74 − 79], as well as gauge quantization [83 − 86] and astrophysical turbulent flows [87], topological charge of vortices in quantum mechanics [88], damping mechanism in planetary physics [89], Landau-Lifschitz equations for ferromagnetism [90, 91]. The results obtained in this work need more details and further consequences are under progress.

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# GEOMETRIJA MNOGOSTRUKOSTI U LIEVOM ENDOMORFSKOM PROSTORU I NJIHOVI PAROVI U FRAKCIJSKOM AKTIVNOM VARIJACIJSKOM PRISTUPU

Raspravljaju se neke zanimljive odlike geometrije mnogostrukosti u Lievom endomorfskom prostoru u frakcijskom aktivnom varijacijskom pristupu (frakcijski diferenciranoj Lagrangeovoj funkciji) kako je to autor nedavno objavio.