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A FRACTIONAL APPROACH TO NONCONSERVATIVE LAGRANGIAN DYNAMICAL SYSTEMS

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In this work, fractional integral calculus is applied in order to derive Lagrangian mechanics of nonconservative systems. In the proposed method, fractional time integral introduces only one parameter, α, while in other models an arbitrary number of fractional parameters (*orders of derivatives*) appears. Some results on Hamiltonian part of mechanics, namely Hamilton equations, are obtained and discussed in detail.

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1. Introduction

We believe today that fractional calculus plays an important role in the understanding of chaotic and scaling behaviours in complex classical and quantum dynamical systems [1 – 10]. It is a quite irreplaceable mean for the description and investigation of such physical process as stochastic and chaotic non-diffusive transport in complex chaotic dynamical systems [11,12]. Fractional Liouville [13], Langevin [14] and Fokker–Planck–Kolmogorov equations [15 – 20] have also been derived and discussed in some details. In fact, the nature of the fractional derivatives or integrals used depends on the specified physical situation [2,5,10,23]. The use of fractional calculus to study nonconservative Lagrangian and Hamiltonian classical dynamical systems has also been investigated in detail [22,24,25]. The applications of the fractional calculus to the constrained dynamical systems and the extension of the fractional variational problem of Lagrange were also treated [26 – 30]. Recently, Lagrangians with linear velocities within Riemann–Liouville fractional derivatives were also treated and investigated [31] and the generalized function approach of nonconservative Lagrangian mechanics was also considered [32]. Dissipative linear dynamical systems with constant coefficients were used to model nonconservative dynamical systems, and have also been discussed by several authors. Some authors claimed that environment and dissipative systems are coupled together [33]. One major problem of this procedure is the introduction of extra coordinates. This difficulty is eliminated when fractional derivatives in the Lagrangians are used [22], but other problem appears, such as the non-causality of equations of motion. Conservative systems in general are simplification of physical reality, since they imply that motion is frictionless even in the presence of strong constraints. The later, as we know, imply the presence of frictional forces, and the motion in a typical physical environment necessarily implies a certain resistance due to the medium. The dissipative forces can be modelled in a huge number of ways. Methodologically, dissipative Newtonian systems are nothing but a complement to the conservative systems, since not only the energy, but also other physical quantity such as linear and angular momentum, are not conserved. In this work, we propose a novel approach to the recent nonconservative models that were studied in the framework of fractional differential calculus. In the proposed method, fractional time integral requires only one parameter, α, while in other models, an arbitrary number of fractional parameters (*orders of derivatives*) appears.

Our paper is organized as follows: In Sec. 2, we derive the fractional Euler–Lagrange equations. In Sec. 3, we give a simple example by applying the resulting fractional Euler–Lagrange equation to analyze the dynamical behaviour of the simple pendulum which undergoes small oscillations. In Sec. 4, we recall that, in Newtonian mechanics, conservation laws can be derived from the first integral Euler–Lagrange equations which seem totally modified using fractional Euler–Lagrange equations. Finally, a short conclusion is given in Sec. 5.

2. *Fractional Euler-Lagrange equations*

Let us consider the left-sided Riemann–Liouville fractional integral of order $\alpha \in [-\infty, +\infty]$ [2,34,35]

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)\big(t-\tau\big)^{\alpha-1} \mathrm{d}\tau. \tag{1} $$

We wish to use this definition and require a procedure for finding the stationary value of the following fractional action

$$ S(q) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t L(\dot{q}, q, \tau)\big(t-\tau\big)^{\alpha-1} \mathrm{d}\tau \tag{2} $$

$$ = \int_{t_0=0}^t L(q, \dot{q}, \tau)\, \mathrm{d}g_t(\tau) \tag{3} $$

where $L(\dot{q}, q, \tau)$ is the Lagrangian weighted with $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$ and $g_t(\tau) = [t^\alpha - (t-\tau)^\alpha]/\Gamma(1+\alpha)$ with the scaling properties $g_{kt}(k\tau) = k^\alpha g_t(\tau)$, $k > 0$. For fixed t, integral (3) becomes a Stieltjes integral [5].

In fact, we consider a typical mechanical system with N degrees of freedom $q = (q_1, \ldots, q_N)$ where we associate a C^2 function of the variables q and $\dot{q}$ and of the time τ, $L(\dot{q}, q, \tau)$. If we let $\phi(\tau) = (\phi_1(\tau), \ldots, \phi_N(\tau))$ in the interval $t_0 \leq \tau \leq t$ be a solution of the equations of motion with fixed boundary values, say $(\phi(t_0), \phi(t)) = (a, b)$, then this solution or orbit is such that the action integral (2) assumes an extreme value [36 − 38]. A necessary condition for the fractional action integral $S(q)$ to assume an extreme value, for $q = \phi(\tau)$, is that $\phi(\tau)$ be an integral curve of the Euler–Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial L}{\partial \dot{q}_k} \equiv -\bar{F}_k, \quad k = 1, \ldots, N. \tag{4}$$

The proof of this statement proceeds as follows: We let $q_0(\tau)$ be the minimum solution and write $q(\tau) = q_0(\tau) + \sigma(\tau)$, where $\sigma(\tau)$ describes the deviation of $q(\tau)$ from the minimum path $q_0(\tau)$ [36,37]. Inserting into Eq. (2) gives

$$S = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} L(\dot{q}_0(\tau) + \dot{\sigma}(\tau), q_0(\tau) + \sigma(\tau), \tau)\big(t-\tau\big)^{\alpha-1}\mathrm{d}\tau. \tag{5}$$

Performing Taylor expansions to first order in $\dot{\sigma}(\tau)$ and $\sigma(\tau)$ yields

$$S = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left[L(\dot{q}_0(\tau), q_0(\tau), \tau) + \sum_k\left\{\frac{\partial L}{\partial \dot{q}_k}\dot{\sigma}_k(\tau) + \frac{\partial L}{\partial q_k}\sigma_k(\tau)\right\}\right]\big(t-\tau\big)^{\alpha-1}\mathrm{d}\tau. \tag{6}$$

We integrate the term in $\sigma(t)$ by parts and then

$$\begin{aligned} S \;=\; & \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} L(\dot{q}_0(\tau), q_0(\tau), \tau)\big(t-\tau\big)^{\alpha-1}\mathrm{d}\tau - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\sum_k \sigma_k(\tau) \\ & \times\left[\big(t-\tau\big)^{\alpha-1}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \big(1-\alpha\big)\frac{\partial L}{\partial \dot{q}_k}\big(t-\tau\big)^{\alpha-2} - \frac{\partial L}{\partial q_k}\big(t-\tau\big)^{\alpha-1}\right]\mathrm{d}\tau. \end{aligned} \tag{7}$$

$\bar{F}_k$ is the modified frictional force. In fact, frictional forces are a common type of non-conservative force. When $\alpha = 1$, we find the standard Euler–Lagrange equation and $\tau \to \infty$, $\bar{F}_k \to 0$. In standard Newtonian mechanics, if one considers an unconstrained Newtonian system with self-adjoint forces, such as the one-dimensional one, then the Lagrangian exists if the fundamental analytic theorem for configurations space formulations is verified. If the system is represented more realistically by adding, for example, a drag force linear in the velocity, the system tends to a non-self-adjoint situation, and as a result, a Lagrangian for its direct representation does not exist. This case, as we will see in the next section, is possible using Eq. (4). A Lagrangian can still exist for the representation of an equivalent equation of motion, provided that it is self-adjoint, under the transformation $\tau = t + T$, $L(q, \dot{q})_\tau \equiv L(q, \dot{q})_T$.

It is apparent than this is no more than a modified Newton's second law of motion written in the q_k coordinate system. The RHS term in Eq. (4) could be viewed as a generalized external force acting on the system. Usually, dissipative forces in Newtonian systems do not affect the degrees of freedom of a physical system. For instance, if dissipative forces are considered for a dynamical system of N particles with $3N - n$ independent holonomic constraints, and thus n degrees of freedom, the number of generalized coordinates remains unchanged. It is then possible to represent dissipative forces in terms of generalized coordinates by considering the virtual work by the dissipative forces, which must be the same in Cartesian and in generalized coordinates [39]. That is, $\{\textit{conservative systems}\} \subset \{\textit{dissipative systems}\}$. Clearly, as we will see in Sec. 3, dissipative systems can also be represented by nonlinear differential equations.

3. Damped oscillatory system

A significant example of a truly one-dimensional dynamic system is a simple pendulum of length l and mass m, attached to the circumference of a body of negligible radius [34]. The linear kinetic energy is therefore $K = (1/2)ml^2\dot{\theta}^2$ and the potential energy for small oscillations is $V = (1/2)mgl\theta^2$. Here θ is the angular coordinate. The Lagrangian is then given by $L = K - V$. Let $T = t - \tau$ be a time-change of variable. Then, making use of Eq. (6), one gets the following differential equation

$$\ddot{\theta} + \frac{\alpha - 1}{T}\dot{\theta} + \omega^2\theta = 0\,, \tag{8}$$

where time-derivative is now done with respect to T and $\omega^2 = g/l$, g being the gravity constant. The second term is the dissipative term with time-decreasing coefficient. This equation can be easily solved if we make the ansatz $\theta = T^\rho\varphi(T)$. This leads to

$$\begin{aligned} &\rho(\rho-1)T^{\rho-2}\varphi + 2\rho T^{\rho-1}\dot{\varphi} + T^\rho\ddot{\varphi} + \frac{\alpha-1}{T}\left(\rho T^{\rho-1}\varphi + T^\rho\dot{\varphi}\right) \\ &\quad = T^{\rho-2}\Big[T^2\ddot{\varphi} + (\alpha - 1 + 2\rho)T\dot{\varphi} + \big(T^2 + \rho(\rho+\alpha-2)\big)\varphi\Big] = 0. \end{aligned} \tag{9}$$

By taking the choice $\alpha + 2\rho = 2$, we get the Bessel's differential equation

$$T^2\ddot{\varphi} + T\dot{\varphi} + \left(T^2 - \rho^2\right)\varphi = 0 \tag{10}$$

of order $\pm\rho$. A system of linearly independent solutions of (10) is given by the pair of Hankel functions $H_\rho^\pm(T)$ so that $\varphi_\pm = T^\rho H_\rho^\pm(T)$ with $\rho = (2-\alpha)/2$ gives a pair of linearly independent solutions of (8).

If we let $T = |\chi|(1+\xi)$, where ξ is a new time variable, than Eq. (8) takes the form (*which is a* $|\chi|$*-independent differential equation*)

$$\theta_{\xi\xi} + \frac{\alpha-1}{1+\xi}\theta_\xi + \left|\chi\right|^2\omega^2\theta = 0. \tag{11}$$

A possible solution is the Fourier image [40]

$$\theta(T,\chi) = F_{x\to\chi}[\tilde{\theta}] = (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} \exp\{-\mathrm{i}x\chi\}\theta(\xi,x)\mathrm{d}x\,, \quad x\in\mathbb{R}^n\,. \tag{12}$$

with $n\equiv\alpha$. Equation (11) is parameterized by the new frequency parameter $\bar{\omega}^2 \equiv |\chi|^2\omega^2$. In fact, Eq. (8) can be written in the following form

$$ml\ddot{\theta} + ml\frac{\alpha-1}{T}\dot{\theta} = -\frac{\partial V}{\partial\theta}\,. \tag{13}$$

This is the causal equation of motion with friction but where the dissipative term is time-decreasing. In reality, fractional operators have memory due to their nonlocality in time. For that purpose, it is crucial to solve the simple pendulum problem with the two time variables $t-\tau$ and $t+\tau$ where τ is the intrinsic time and t is the observer's time. If we repeat the same calculation steps, one finds

$$ml\ddot{\theta} + ml\frac{\alpha-1}{T_{\leftarrow}}\dot{\theta} = -\frac{\partial V}{\partial\theta}\,, \qquad \text{(Retarded)} \tag{14}$$

$$ml\ddot{\theta} - ml\frac{\alpha-1}{T_{\rightarrow}}\dot{\theta} = -\frac{\partial V}{\partial\theta}\,, \qquad \text{(Advanced)} \tag{15}$$

where $T_{\leftarrow}$ and $T_{\rightarrow}$ are new notations for retarded and advanced time-change variables. Allowing both a retarded and an advanced equation of motion to arise from the fractional action, fractional Euler–Lagrange equation seems natural. In this way, fractional action is preferable because it does not assume *a priori* the left fractional derivatives to be favored over the right fractional ones. Now we turn our attention to the Hamiltonian equations of motion.

4. *The Hamiltonian formulation of dynamics*

In classical mechanics, conservations laws are derived from the first integral of the Euler–Lagrange equations. Certainly, the presence of the $(1-\alpha)/(t-\tau)\partial L/\partial\dot{q}_i$

in the RHS of Eq. (4) will change the situation. Let us first discuss the case where the Lagrangian is not a function of q_i, that is $\partial L/\partial q_i = 0$. Hence, Eq. (4) reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\alpha - 1}{t - \tau}\,\frac{\partial L}{\partial \dot{q}_i}\,, \tag{16}$$

giving $\partial L/\partial \dot{q}_i \propto (\tau - t)^{1-\alpha}$. Consider now the motion of a particle in the absence of force. Then $L = (1/2)m\dot{\vec{r}}^{\,2}$ and $\partial L/\partial \dot{q}_i = \partial L/\partial \dot{\vec{r}} = m\dot{\vec{r}} \propto (\tau - t)^{1-\alpha}$. It is clear that this result represent a deviation from the Newton's first law of motion. Only when $\alpha = 1$, that is, when no fractional integral is considered, that we fall into the standard Newton's first law of motion. For an arbitrary coordinate system, we can define a fractional conjugate momentum as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \propto \big(\tau - t\big)^{1-\alpha}. \tag{17}$$

This is an anomalous conjugate momentum and it does have the property that, if L does not depend on q_i, it is not a constant of motion.

In general, if the Lagrangian function has no explicit time dependence, then the function

$$H(q,\dot{q}) = \sum_{k=1}^{N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L(q,\dot{q}) \tag{18}$$

must be is a constant of motion. Indeed, differentiating with respect to time gives

$$\frac{\mathrm{d}H}{\mathrm{d}\tau} = \sum_{i=1}^{N}\left[\ddot{q}_i\frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] \tag{19}$$

$$= \sum_{i=1}^{N}\left[C\dot{q}_i\big(1-\alpha\big)\big(\tau - t\big)^{-\alpha} - \frac{\partial L}{\partial q_i}\dot{q}_i\right] \tag{20}$$

$$= \sum_{i=1}^{N}\left[\dot{q}_i\big(1-\alpha\big)\big(\tau - t\big)^{-\alpha}\right] \equiv \frac{1-\alpha}{(\tau - t)^{\alpha}}\sum_{i=1}^{N}\dot{q}_i, \tag{21}$$

where $C = 1$ in Eq. (21) for simplicity, and use was made of $\delta L/\delta q_k = 0$. One can remark that the energy is conserved only when $\alpha = 1$. As $\tau \to \infty$, that is at late times, $\mathrm{d}H/\mathrm{d}\tau \to 0$. In fact in classical mechanics, Hamiltonian given by Lagrangian not depending on time is a constant of motion. In fractional mechanics this is not the case.

It is possible to obtain the fractional canonical equations directly from our fractional Hamiltonian's variational principle by writing

$$F(q,p,\dot{q},\dot{p},\tau) = \sum_{k=1}^{N} p_k\dot{q}_k - H(q,p,\tau) \tag{22}$$

where $q, p, \dot{q}, \dot{p}$ are four sets of independent variables. Requiring that

$$\delta\left\{\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} F(q,p,\dot{q},\dot{p},\tau)\big(t-\tau\big)^{\alpha-1}\mathrm{d}\tau\right\} = 0\,, \tag{23}$$

and varying the variables q_k and p_k independently, we get our generalized Euler–Lagrange equations $\delta F/\delta q_k = 0$, $\delta F/\delta p_k = 0$. When written, they are

$$\frac{\partial F}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial F}{\partial \dot{q}_k}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial F}{\partial \dot{q}_k}\,, \tag{24}$$

$$\frac{\partial F}{\partial p_k} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial F}{\partial \dot{p}_k}\right) = \frac{1-\alpha}{t-\tau}\frac{\partial F}{\partial \dot{p}_k}\,, \tag{25}$$

or

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} - \frac{1-\alpha}{t-\tau}p_k\,, \tag{26}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k}\,, \tag{27}$$

$k = 1, \ldots, N$. This pair of equations represents the fractional canonical equations. In fact, the presence of the second term on the RHS of Eq. (26) is not a surprise. Remember that the Euler–Lagrange equations with external forces are

$$\frac{\partial F}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial F}{\partial \dot{q}_k}\right) = -F_k, \tag{28}$$

and the corresponding Hamilton's equations are

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} + F_k, \tag{29}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k}. \tag{30}$$

We note that Hamilton's equations with external forces violate the Lie algebra conditions and verify instead the Lie-admissible algebra [39].

The last comment concerns the Poisson bracket for f and g. It is in fact a skew-symmetric bilinear of derivatives of any pair of dynamical quantities with respect to coordinates and momenta. A Poisson bracket is defined by

$$\{f,g\}(x) = \sum_{i=1}^{N}\left\{\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right\}. \tag{31}$$

It is possible to write, in a more symmetric form, the canonical Eqs. (26) and (27) by means of Poisson brackets. Indeed, one can easily verify that they read

$$\dot{p}_k + \frac{\alpha-1}{t-\tau} p_k = \{H, p_k\}, \tag{32}$$

$$\dot{q}_k = \{H, q_k\}. \tag{33}$$

If we let $g(q,p,\tau)$ be a dynamical quantity, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau} g(q,p,\tau) = \frac{\partial g}{\partial \tau} + \sum_{k=1}^{N} \left(\frac{\partial g}{\partial q_k}\dot{q}_k + \frac{\partial g}{\partial p_k}\dot{p}_k \right) \tag{34}$$

$$= \frac{\partial g}{\partial \tau} + \sum_{k=1}^{N} \left(\frac{\partial g}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k}\left(\frac{\partial H}{\partial q_k} + \frac{\alpha-1}{\tau-t} p_k \right) \right) \tag{35}$$

$$= \frac{\partial g}{\partial \tau} + \{H, g\} - \frac{\alpha-1}{\tau-t} \sum_{k=1}^{N} p_k \frac{\partial g}{\partial p_k}\,. \tag{36}$$

If g is an integral of motion, then

$$\frac{\partial g}{\partial \tau} + \{H, g\} = \frac{\alpha-1}{\tau-t} \sum_{k=1}^{N} p_k \frac{\partial g}{\partial p_k}\,, \tag{37}$$

and if g has no explicit time dependence

$$\{H, g\} = \frac{\alpha-1}{\tau-t} \sum_{k=1}^{N} p_k \frac{\partial g}{\partial p_k}\,. \tag{38}$$

As a result, we have noncommutativity. Commutativity occurs at late times or for $\alpha = 1$.

5. Conclusions

In this paper, we use left-sided Riemann–Liouville fractional integral to write the Euler–Lagrange equations in fractional form. The derived equations are similar to the standard ones, but with the presence of fractional generalized external force acting on the system. As an application, we studied the simple pendulum. By treating the action as a fractional integral, the presence of a linear time-decreasing dissipative term force is shown. By introducing two time variables, we derived the natural advanced and retarded equations of motion. Hamiltonian conservative laws are also treated. The conjugate momentum, the Hamiltonian and the Hamilton's equations are shown to depend on the fractional order of integration α and vary as inverse of time. We hope that our approach will open the possibility of examining deeply theoretical mechanics and complex dynamical systems. Further consequences and developments, in particular the inverse problem in fractional Newtonian mechanics, relativistic dynamics, geometrical aspects and symplectic manifolds as well as some applications to quantum mechanics and fields theory will be dealt with in future work.

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NECJELOBROJNA HAMILTONOVA VARIJACIJSKA FORMULACIJA NEKONZERVATIVNE LAGRANGEOVE MEHANIKE

U ovom se radu primjenjuje necjelobrojni integralni račun radi izvođenja Lagrangeove mehanike nekonzervativnih sustava. U predloženoj se metodi uvodi necjelobrojni vremenski integral samo s jednim parametrom, α, dok u drugim modelima nalazimo proizvoljne brojeve necjelobrojnih parametara (*redova derivacija*). Neki rezultati Hamiltonove mehanike, naime Hamiltonove jednadžbe, izvedeni su i raspravljaju se podrobno.