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EINSTEIN'S LIGHT COMPLEX

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The light complex introduced by A. Einstein in his first relativity paper has the characteristics of a provisional concept and was soon abandoned. Nevertheless, it was crucial for Einstein's derivation of the equivalence of mass and energy. The discussion of various forms of the light complex strictly within Maxwell's electrodynamics leads to interesting insight.

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1. Introduction

As the centenary of Einstein's *annus mirabilis*, linked with the *World Year of PHYSICS 2005*, is approaching, his seminal work is attracting again attention. Among Einstein's papers of 1905 *On the Electrodynamics of Moving Bodies* [1] and its sequel *Does the inertia of a body depend upon its energy content?* [2] may deserve special interest. In these papers, with one stroke, the theory of relativity was presented, to become the special theory of relativity. Remarkably, all pieces fell to their places and later on nothing had to be amended and little added.

In the first paper Einstein introduced the concept of the light complex (Lichtkomplex). The equation derived by means of it was crucial in the subsequent derivation of "the world's most famous equation", $E = mc^2$ [2, 3].

Apparently, Einstein did not use the term "light complex" any more and even in the sequel he preferred the term "quantity of light" (Lichtmenge). In an overview, he derived only the equations (1) for the field, not mentioning its energy [4]. The light complex has all the characteristics of a provisional concept, introduced in the context of discovery, to be abandoned later on. As such, it was not often discussed [5, 6]. So a discussion of the light complex from the historical and didactical point of view may be worthwhile.

We start by introducing a plane light complex which in a very simple way leads to Einstein's result. We refine the concept by introducing a prismatic light complex and later on thoroughly calculate its energy. Although for the Einstein's spherical light complex this cannot be done directly, the same result applies. A few concluding remarks are added.

2. A plane light complex

The simplest possible light complex is considered and the simplest way to Einstein's relation chosen without loss of generality. The units used by Einstein are replaced by SI units and contemporary symbols are used making the work more readable.

Consider in an inertial frame of reference S with axes x, y, z a monochromatic linearly polarized plane wave propagating in the direction $(\cos\phi, \sin\phi, 0)$ with the electric and magnetic field $\boldsymbol{\mathcal{E}} = (-\mathcal{E}\sin\phi, \mathcal{E}\cos\phi, 0)$ and $\boldsymbol{\mathcal{B}} = (0, 0, \mathcal{E}/c)$, respectively. $\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}} - c\boldsymbol{\mathcal{B}}\cdot c\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{B}}$ are invariants and $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{B}}\times\boldsymbol{c}$ wherein $\boldsymbol{c}$ is the velocity of light. Using the transformation equations [1], the field components in the reference frame S′ moving with velocity v along the common x and x'-axes are obtained

$$\begin{aligned} \mathcal{E}'_x &= -\mathcal{E}'\sin\phi' = \mathcal{E}_x = -\mathcal{E}\sin\phi \\ \mathcal{E}'_y &= \gamma(\mathcal{E}_y - v\mathcal{B}_z) = \gamma\mathcal{E}(\cos\phi - v/c) \\ c\mathcal{B}'_z &= \gamma(c\mathcal{B}_z - v\mathcal{E}_y/c) = \gamma c\mathcal{B}(1 - v\cos\phi/c)\,, \end{aligned} \tag{1}$$

whereas $\mathcal{E}'_z$, $\mathcal{B}'_x$, and $\mathcal{B}'_y$ equal zero, and $\gamma = (1 - v^2/c^2)^{-1/2}$.

The energy density of the electromagnetic field in the frame S is given by $w = \frac{1}{2}\varepsilon_0\mathcal{E}^2 + \frac{1}{2}\mathcal{B}^2/\mu_0$. The energy density in the frame S′ can be obtained by the stated transformation equations with the energy density in S

$$w' = \frac{\mathcal{E}'^2_x + \mathcal{E}'^2_y + c^2\mathcal{B}'^2_z}{\mathcal{E}^2_x + \mathcal{E}^2_y + c^2\mathcal{B}^2_z} w = \gamma^2(1 - v\cos\phi/c)^2 w\,. \tag{2}$$

The time average of the energy density in the S frame $\overline{w} = \frac{1}{4}\varepsilon_0(\mathcal{E}^2_{x0} + \mathcal{E}^2_{y0}) + \frac{1}{4}\mathcal{B}^2_0/\mu_0 = \frac{1}{2}\varepsilon_0\mathcal{E}^2_0$ is taken over a number of periods T with the field amplitudes $\mathcal{E}_0$ and $\mathcal{B}_0$. The time average of the energy density $\overline{w}'$ in the S′ frame is taken correspondingly over a number of periods $T' = \lambda'/c = \lambda/[\gamma(1 - v\cos\phi/c)]c$. Equations connecting the angles ϕ

$$\cos\phi' = \frac{\cos\phi - v/c}{1 - v\cos\phi/c}\ , \quad \cos\phi = \frac{\cos\phi' + v/c}{1 + v\cos\phi'/c} \tag{3}$$

and

$$\sin\phi' = \frac{\sin\phi}{\gamma(1 - v\cos\phi/c)}\,, \quad \sin\phi = \frac{\sin\phi'}{\gamma(1 + v\cos\phi'/c)}$$

will be useful. The first two are derived from the components of the electric field in both frames and the second two follow from the relativity principle.

Let us assume a plane light complex between two planes propagating in the reference frame S according to

$$x_f = ct\cos\phi \qquad \text{and} \qquad x_b = c(t-t_1)\cos\phi\,, \tag{4}$$

where x_f and x_b are the x-coordinates of the points in the front and back planes, respectively. To consider that plane light complex in the reference frame S′, we use the inverse Lorentz transformation,

$$x = \gamma(x' + vt')\,, \quad y = y'\,, \quad z = z'\,, \quad t = \gamma(t' + vx'/c^2)$$

as Einstein used to do [1]. Equations (4) transform into

$$x'_f = ct'\cos\phi' \qquad \text{and} \qquad x'_b = ct'\cos\phi' - ct_1\cos\phi/[\gamma(1 - v\cos\phi/c)]\,. \tag{5}$$

The quotient of volumes of the complex in S′ and S is

$$\frac{V'}{V} = \frac{x'_f - x'_b}{x_f - x_b} = \frac{1}{\gamma(1 - v\cos\phi/c)}\,. \tag{6}$$

The energy of the light complex in the S frame is given by $E = \overline{w}V$ and in the frame S′ by $E' = \overline{w}'V'$. So the energies of the complex are connected by

$$E' = \frac{\overline{w}'V'}{\overline{w}V}\,E = \gamma(1 - v\cos\phi/c)E\,. \tag{7}$$

This relation was Einstein's basis for his derivation of the equivalence of mass energy [2].

3. A prismatic light complex

The plane light complex (4) is unlimited in the direction of the y and z axes. If we consider a prismatic light complex, this shortcoming is avoided. Assume an aperture extending in the plane $x = 0$ between $-\frac{1}{2}b \le y \le \frac{1}{2}b$ and $-\frac{1}{2}b \le z \le \frac{1}{2}b$. It is provided with a shutter which by the observer in S is opened from $t = 0$ to $t = t_1$. The shutter is considered a mnemonic device to define a light complex in the form of a prism in an unperturbed plane wave. This is in accord with Einstein who introduced the boundary of the light complex as a surface through which no energy is flowing.

The sides of the prismatic light complex are given by the equations

$$y_f = ct\sin\phi \pm \tfrac{1}{2}b\,, \qquad y_b = c(t-t_1)\sin\phi \pm \tfrac{1}{2}b \quad \text{and} \quad z = \pm\tfrac{1}{2}b \tag{8}$$

together with equations (4). By the inverse transformation and the equations (5) with $x'_b = c(t' - t'_1)\cos\phi' - vt'_1$ we obtain

$$y'_f = ct'\sin\phi' \pm \tfrac{1}{2}b\,, \qquad y'_b = c(t' - t'_1)\sin\phi' \pm \tfrac{1}{2}b, \qquad \text{where} \quad t'_1 = \gamma t_1\,. \tag{9}$$

The prisms have laterally a pair of boundary planes perpendicular to the z and z'-axes and a pair of boundary planes inclined by ϕ towards the x-axis in S and by $\tilde{\phi}'$ towards the x'-axis in S$'$. Besides, there are two basic boundary planes perpendicular to the x and x' axis. The definitions and equations (3) lead to the transformation

$$\tan\phi = \frac{y_f - y_b}{x_f - x_b}\,, \qquad \tan\tilde{\phi}' = \frac{y'_f - y'_b}{x'_f - x'_b}\,, \qquad \tan\tilde{\phi}' = \frac{\sin\phi'}{\cos\phi' + v/c} = \gamma\tan\phi\,. \tag{10}$$

The prismatic light complex in S$'$ is propagating in the direction given by the angle ϕ', which is different from the angle $\tilde{\phi}'$. However, the ratio of volumes of the complex in S$'$ and in S and equation (7) remain valid.

4. *The energy of a prismatic light complex in detail*

One could object to the general idea that the energy of a light complex was obtained by multiplying the *time* average of the energy density by the *volume*. So let us discuss the transformation of the energy of a prismatic light complex in some detail. In an inertial frame, the energy of a light complex equals, by definition, the integral of the electromagnetic energy density over the volume of the complex at a fixed time. At all points, a light complex is travelling with the same velocity which is the phase velocity of the plane wave from which the complex is "cut". So the phase of the electric and magnetic field is "frozen" at a moving point of the complex and the energy of the complex is time-independent.

The energy of the prismatic light complex in the S frame can be obtained in two ways. First, it can be computed as the integral of the electromagnetic energy density over the volume of the complex at the time, say, $t = 0$. Second, it can be obtained as the energy of the plane wave transmitted through the aperture in the plane $x = 0$, during the time interval from $t = 0$ to $t = t_1$. Since the transmitted energy per second equals the flux of the x-component of the Poynting vector $\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}/\mu_0$ through the aperture, it follows that the energy of the prismatic complex in the S frame equals

$$E = \int\limits_0^{t_1} \int\limits_{-b/2}^{b/2} \int\limits_{-b/2}^{b/2} w(x = 0, y, z, t)\mathrm{d}t\mathrm{d}y\mathrm{d}z c\cos\phi =$$

$$= \int\limits_{-b/2}^{b/2} \int\limits_{-b/2}^{b/2} \overline{w} t_1 \mathrm{d}y\mathrm{d}z c\cos\phi = \overline{w}V\,. \tag{11}$$

The second method for calculating the energy of the prismatic complex is valid because, in S, the light prism propagates in a direction given by the angle ϕ which is parallel to two lateral boundary planes of the prism the aperture being at rest in S.

In the S′ frame the aperture is moving and lateral boundary planes make the angle $\tilde{\phi}'$ with the x' axis which is different from the angle ϕ' giving the direction of propagation. As a consequence, the second method for obtaining the energy of the prismatic complex does not apply in S′ because the light "pencils" of which the prism is composed, while propagating in the direction of ϕ', have various lengths. Therefore, one must calculate the energy of the complex following its definition, as the volume integral of the electromagnetic energy density.

Due to the simple geometry, the calculation by the second method can be done in both frames. In the frame S, the electromagnetic energy density of the plane linearly polarized wave propagating in the direction $(\cos\phi, \sin\phi, 0)$ is given by

$$w(\boldsymbol{r}, t) = \varepsilon_0 \mathcal{E}_0^2 \cos^2[2\pi t/T - 2\pi(x\cos\phi + y\sin\phi)/\lambda]\,. \tag{12}$$

The energy of the prismatic light complex introduced above, calculated at time $t = 0$, is

$$E = \int\limits_{-ct_1\cos\phi}^{0} \int\limits_{y_1(x)}^{y_2(x)} w(x, y, t=0) b\,\mathrm{d}y\,\mathrm{d}x, \tag{13}$$

where $y_2(x) = x\tan\phi + \frac{1}{2}b$ and $y_1(x) = x\tan\phi - \frac{1}{2}b$ are the equations of the lateral boundary planes of the prism in S. Equations (12) and (13) give

$$E = \int\limits_{-ct_1\cos\phi}^{0} b\varepsilon_0 \mathcal{E}_0^2 \mathrm{d}x \int\limits_{y_1(x)}^{y_2(x)} \cos^2[(2\pi x\cos\phi)/\lambda + (2\pi y\sin\phi)/\lambda]\mathrm{d}y = \overline{w} V F(\phi, \lambda, t_1)\,,$$

$$F(\phi, \lambda, t_1) = \left[1 + \mathrm{sinc}\left(\frac{2\pi}{\lambda} b\sin\phi\right) \mathrm{sinc}\left(\frac{4\pi}{\lambda} ct_1\right)\right] \tag{14}$$

with the volume of the complex $V = b^2 ct_1 \cos\phi$ and $\mathrm{sinc}\,\xi \equiv (\sin\xi)/\xi$.

In a similar way, one obtains the energy E' of the prismatic light complex in the S′ frame at $t' = 0$,

$$E' = \int\limits_{-ct_1'(\cos\phi' + v/c)}^{0} b\varepsilon_0 \mathcal{E}_0'^2 \mathrm{d}x' \int\limits_{y_1'(x')}^{y_2'(x')} \cos^2[(2\pi x'\cos\phi')/\lambda' + (2\pi y'\sin\phi')/\lambda']\mathrm{d}y'\,, \tag{15}$$

where $y_2'(x') = x'(\sin\phi')/(\cos\phi' + v/c) + \frac{1}{2}b$ and $y_1'(x') = x'(\sin\phi')/(\cos\phi' + v/c) - \frac{1}{2}b$ are the equations of the boundary planes of the prism in S′. The final result for E' reads

$$E' = \overline{w}' V' F(\phi, \lambda, t_1)\,, \tag{16}$$

where $V' = b^2 ct'(\cos\phi' + v/c)$ is the volume of the complex in S′. At first sight, one is somewhat surprised to find that the function $F(\phi, \lambda, t_1)$ is a Lorentz scalar. Thus equation (7) is exactly valid.

5. The energy of Einstein's spherical light complex

Let us now turn to Einstein's light complex that has a spherical shape in the S frame. According to the argument in the preceding paragraph, it follows that the energy of the spherical complex must be calculated as the volume integral of the energy density. Consider a spherical complex of radius R "cut" in the plane wave introduced above, whose electromagnetic energy density is given by Eq. (12). The equation of the spherical surface moving at the phase velocity of the wave is given by

$$(x - ct\cos\phi)^2 + (y - ct\sin\phi)^2 + z^2 = R^2\,, \tag{17}$$

assuming the center of the sphere is at the origin at the time $t = 0$. The energy of the spherical complex calculated at $t = 0$ is

$$E = \varepsilon_0 \mathcal{E}_0^2 \int \cos^2[(2\pi x\cos\phi)/\lambda + (2\pi y\sin\phi)/\lambda]\mathrm{d}V\,, \tag{18}$$

where the integration extends over the volume of the sphere. After a new coordinate system with a polar axis coinciding with the direction of wave propagation is introduced, the integration is performed in spherical coordinates

$$E = 2\pi\varepsilon_0\mathcal{E}_0^2 \int_0^{\pi}\int_0^{R} \cos^2\left(2\pi r\cos\theta/\lambda\right) r^2 \sin\theta \mathrm{d}\theta\mathrm{d}r = \overline{w} V F(\lambda/R)\,,$$

$$F(\lambda/R) = 1 + 3\left(\frac{4\pi R}{\lambda}\right)^{-2}\left[\mathrm{sinc}\left(\frac{4\pi R}{\lambda}\right) - \cos\left(\frac{4\pi R}{\lambda}\right)\right]\,, \tag{19}$$

where $V = \frac{4}{3}\pi R^3$ is the volume of the complex. Note that $F \to 2$ in the limit $R \to 0$ for an infinitesimal light complex, since the phase of the wave we consider is zero at the origin.

In the S′ frame, the surface of the spherical light complex is an ellipsoidal surface whose equation at the time $t' = 0$ reads

$$\gamma^2 x'^2(1 - 2(v/c)\cos\phi + (v/c)^2) + y'^2 - 2x'y'\gamma(v/c)\sin\phi + z'^2 = R^2. \tag{20}$$

The volume of the ellipsoid

$$V' = \tfrac{4}{3}\pi R\sqrt{(1+v/c)/(1-v\cos\phi/c)}\cdot R\sqrt{(1-v/c)/(1-v\cos\phi/c)}\cdot R =$$
$$= V/[\gamma(1 - v\cos\phi/c)]$$

is connected to the volume of the corresponding sphere V by equation (6).
We refrain from calculating directly the energy E' of the ellipsoidal complex in the S' frame.

6. A look back

After M. Planck [7, 8] in 1906 introduced the relativistic momentum and the transformation for its components, the energy of a light complex could be obtained directly by

$$E' = \gamma(E - vP_x) = \gamma(E - vE\cos\phi/c) = \gamma(1 - v\cos\phi/c)E\,. \tag{21}$$

The relation of the energy of electromagnetic waves E and the magnitude of momentum P, $E = cP$, was supplied by Maxwell already. Accepting this, the concept of the light complex becomes obsolete.

Nowadays, the energy-momentum tensor of the electromagnetic field $T^{\mu\nu}$ is introduced with the energy density $T^{00} = w$. The transformation equation for it is derived most easily by forming the tensor product of two transformed four-vectors giving for the 00-element

$$T^{00\prime} = \gamma^2(T^{00} - 2(v/c)T^{01} + (v/c)^2T^{11})\,.$$

In our case $T^{0i} = T^{i0} = c^{-1}\,\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}/\mu_0 = \varepsilon_0\mathcal{E}^2(\cos\phi, \sin\phi, 0)$ and $T^{11} = \frac{1}{2}\varepsilon_0(-\mathcal{E}_x^2 + \mathcal{E}_y^2 + \mathcal{E}_z^2 - (c\mathcal{B}_x)^2 + (c\mathcal{B}_y)^2 + (c\mathcal{B}_z)^2) = \varepsilon_0\mathcal{E}^2\cos^2\phi$, so equation (2) is recovered [9].

As $\int T^{0\nu}\mathrm{d}V$ is a four-vector "the momentum and energy of a radiation pulse totally contained within a finite volume has the same transformation properties as a material point particle" [10]. The conclusion is generally valid, also for a spherical light complex. In particular, for the prismatic light complex it vindicates the way to equation (16).

We have defined the time average of the energy density over a number of periods of the electromagnetic wave. If we would, alternatively, define it over a time interval $t_1 \to \infty$ the function $F(\phi, \lambda, t_1)$ would tend towards 1. The same would be the case with $F(\lambda/R)$ for $R \to \infty$. In this case it is evident that equation (7) is valid at least to a good approximation.[1] Further, in this case the shutter could be thought of as an idealized apparatus and the light complex a wave packet within the approximation of geometric optics demanding also $b \gg cT$.

What was described in the last section came after 1905. Einstein in Ref. [1], being content with the volume he obtained for the spherical light complex in S′, which led him directly to equation (7), probably did not ask the question we have studied in sections 4 and 5. In any case, he anticipated the scalar character of the function F and the tensorial character of the energy-momentum of the electromagnetic field.

Note added in proof. After deriving equation (7) Einstein remarked: "It is noteworthy that the energy and the frequency of a light complex vary with the observer's state of motion according to the same law." [1] This is an allusion to the light quantum introduced in his earlier paper concerning the production and transformation of light. It would be tempting to assume that the light complex was

[1] Note that for a light complex "cut" in a *circularly* polarized plane wave equation (7) is always exactly valid since the corresponding energy densities are space and time-independent.

some classical counterpart of the quantum [5]. It should be remembered, however, that Einstein at that time has not yet introduced the linear momentum of the quantum. Anyhow, he was well aware that the connection of the two concepts is not a simple one, as according to his biographer A. Pais "it was Einstein's style to avoid the quantum theory if he could help it". The present authors share this point of view. We are indebted to Professor John Stachel for suggesting us to make this point.

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EINSTEINOV SVJETLOSNI KOMPLEKS

Svjetlosni kompleks koji je A. Einstein uveo u svom prvom članku o teoriji relativnosti ima karakteristike privremenog koncepta i nakon toga nije se više upotrebljavao. Bez obzira na to bio je od presudnog značaja za izvodjenje ekvivalencije mase i energije. Diskusija različitih oblika svjetlosnog kompleksa dovodi do zanimljivih uvida.