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# NON-LINEAR PROPAGATION OF WAVES IN A WARM MAGNETIZED PLASMA

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Non-linear propagation of finite-amplitude ion-cyclotron waves, ion-acoustic waves and solitary waves in a collisionless magnetized plasma, consisting of warm ions and isothermal electrons, is theoretically studied. New analytical solutions have been obtained for the excitation of these waves in the plasma. The nature of solutions for these waves are represented graphically. It is seen that the normalized electric field can have arbitrary small as well as large values for the existence of ion-cyclotron waves which contradicts the results of other authors.

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## 1. *Introduction*

In last few years, the excitation and propagation of ion-acoustic solitary waves has been a topic of considerable interest both in laboratory and space plasmas. Washimi and Tanuiti [1] were the first to derive the Korteweg-deVries (K-dV) equation for the ion-acoustic solitary wave in a plasma. Subsequently, incorporating various parameters in the plasma, many authors have studied theoretically [2 − 11] and experimentally [12 − 15] the propagation of ion-acoustic solitons and have obtained many important results. The ion-acoustic wave in a magnetized plasma shows interesting characteristics as there exist two low-frequency electrostatic modes, namely, the ion-acoustic mode and ion-cyclotron mode. Zhakharov and Kuznetsov [16] introduced non-linear wave equation for a magnetized plasma and showed that small-amplitude three-dimensional ion-acoustic solitons can exist in a low-$\beta$ magnetized plasma. In kinetic approach. Swift [17] solved the Poission-Vlasov equations for a magnetized plasma by calculating the ion-density from adiabatic theory. Shukla and Yu [18] showed that the finite-amplitude ion-acoustic solitons propagating obliquely to an external magnetic field can occur in a plasma. Lee and Kan [19] presented a unified formulation for the study of non-linear low-frequency electrostatic wave in a magnetized plasma having cold ions and warm electrons. Besides these, the propagation of ion-acoustic solitons in a magnetized plasma have been studied theoretically [20 – 27] and experimentally [28 – 30] by other authors and found some fascinating results which have wide physical applications.

It is to be mentioned that many authors have used different mathematical techniques for finding the solution of non-linear evolution equation in a magnetized plasma and each of them obtained interesting results. But still, there is enough scope for developing the theoretical infrastructure for the ion-acoustic waves and solitons in plasmas. With this motivation, we have studied in the present paper the non-linear propagation of finite-amplitude ion-cyclotron waves, ion-acoustic waves and solitons in a warm magnetized plasma and have found new analytical solutions for the excitation of these waves. It is seen that the normalized electric field can have arbitrary small as well as large values for the existence of ion-cyclotron waves which contradicts the results obtained by Yashvir et al. [31]. However, some results of Yashvir et al. [31] are verified by our present analytical study.

## 2. Formulation

We consider an isothermal plasma having warm ions and electrons. We assume that the plasma is collisionless and magnetized. The effect of Landau damping is neglected here. Moreover, we assume that the velocities of ions and electrons are non-relativistic. Therefore, the basic equations governing the dynamics of the plasma can be written as:

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0 \,, \tag{1}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\, \vec{u} = -\frac{\vec{\nabla} p}{mn} - \frac{e}{m}\, \vec{\nabla}\phi + \vec{u} \times \vec{\Omega} \,, \tag{2}$$

$$\nabla^2 \phi = n_e - n \,, \tag{3}$$

where $n_e = n_0 \exp(e\phi/T_e)$, $n_e$ and $n$ are, respectively, the density of electrons and ions, $m$ and $\vec{u}$ are the mass and velocity of ions, $\phi$ is the electrostatic potential, $\Omega$ is the gyro-frequency, $e$ is the magnitude of electronic charge, $T_e$ is the electron temperature and $n_0$ is the equilibrium density of electrons and ions.

Now, transforming Eqs. (1) – (3) to non-dimensional form and following the usual procedure [31], we obtain the evolution equation in a new single variable S,

$$\frac{\mathrm{d}F(N)}{\mathrm{d}S^2} = G(N)\,, \tag{4}$$

where

$$F(N) = \ln N + \frac{M^2}{2N^2} + \frac{\sigma N^2}{2}\,,$$

$$G(N) = (N-1)(M^2 - \gamma^2 N) - \frac{\gamma N^2 \sigma}{3}(N^3 - 1)\,, \tag{5}$$

$$N = \frac{n}{n_0} = N(S)\,, \qquad \alpha^2 + \gamma^2 = 1\,, \qquad \sigma = 3T_i/T_e\,, \qquad M = \frac{V_P}{C_S}\,,$$

$T_i$ is the ion temperature, $C_S$ is the speed of ion sound, $\alpha = K_x/K = \sin\theta$, $\gamma = K_z/K = \cos\theta$ and $\theta$ is the angle between the wave vector $K$ and magnetic field $B$. The wave vector $K$ is taken in $(x-z)$ plane. $V_P\ (= w/K)$ is the wave speed and $w$ is the wave frequency.

## 3. *Analytical solutions*

Yashvir et al. [31] obtained three types of real solutions of equation (4), which correspond finite amplitude ion-cyclotron waves, ion-acoustic waves and solitons. However, their calculations involved considerable numerical work using approximations. In the present section, we intend to find the solutions of Eq. (4) by a more rigorous analytical study and to verify the results of the above authors.

To find the solutions of Eq. (4), we first integrate it and obtain

$$\left(\frac{\mathrm{d}F}{\mathrm{d}N}\right)^2 \left(\frac{\mathrm{d}N}{\mathrm{d}S}\right)^2 = H(N) - C\,, \tag{6}$$

where

$$H(N) = 2\int F'(N)G(N)\mathrm{d}N\,, \tag{7}$$

and $C$ is an arbitrary constant. In order to have physically admissible solutions,

(i) $\left(\frac{\mathrm{d}F}{\mathrm{d}N}\right)^2 \left(\frac{\mathrm{d}N}{\mathrm{d}S}\right)^2$ must be non-negative,

(ii) $N$ must be bounded, and

(iii) $N$ must admit unperturbed value 1.

So, for physical solution, the requirements are stated as follows.

**Requirement 1**. There exist $N_1$ and $N_2$ such that

$$N_1 < N_2 \qquad \text{and} \qquad H(N_1) = H(N_2) = C.$$

**Requirement 2**. $H(N) \geq C$ for $N \in [N_1, N_2]$.

**Requirement 3**. $1 \in [N_1, N_2]$.

From the expressions for $F$ and $G$ in Eq. (5), we make the following observations.

**Observation 1**.

For $N > 0$, $G(N) = 0$ holds at two points; one is $N = 1$ and other $N = N'$ satisfying the inequality $N' \lesseqgtr 1$ according to $2(\sigma+1) \lesseqgtr M^2$.

**Observation 2**.

For $N > 0$, $F'(N) = 0$ can hold at only one point $N_0$, such that $\sigma + 1 \lesseqgtr M^2$ according to $N_0 \lesseqgtr 1$. Proof of the observations 1 and 2 are given in Appendix.

Now we consider different cases.

**Case 1a**.

$$M^2 > \sigma + 1, \quad 1 < N' < N_0\,. \tag{8}$$

**Theorem 1**. For $M^2 > \sigma + 1$ and $1 < N' < N_0$, the existence of physical solution of Eq. (3) requires $N_1 < 1 < N_2 < N'$.

**Proof**. From Requirement 3, we need

$$N_1 \leq 1 \leq N_2\,. \tag{9}$$

From Requirement 1 and Requirement 2, we need

$$H'(N_1) \geq 0 \qquad \text{and} \qquad H'(N_2) \leq 0\,. \tag{10}$$

Therefore, from (10) and (7), we need

$$F'(N_1)G(N_1) \geq 0\,, \qquad\qquad F'(N_2)G(N_2) \leq 0\,. \tag{11}$$

Since $F'(N_0) = 0$, we get from (5)

$$F'(N) \gtrless 0 \qquad \text{for} \qquad N \gtrless N_0\,. \tag{12}$$

Also, since $G(1) = G(N') = 0$, we get from (5)

$$G(N) \leq 0 \text{ for } N < 1, \qquad G(N) > 0 \text{ for } 1 < N < N', \qquad G(N) < 0 \text{ for } N' < N\,. \tag{13}$$

From (12) and (13), one can write

$$F'(N)G(N) > 0 \text{ for } \; N < 1, \qquad F'(N)G(N) < 0 \; \text{ for } \; 1 < N < N', \tag{14}$$
$$F'(N)G(N) > 0 \text{ for } \; N' < N < N_0, \qquad F'(N)G(N) < 0 \; \text{ for } \; N > N_0.$$

From (8), (9), (11) and (14), it follows that

$M^2 > \sigma{+}1$ for either $N_1 < 1 < N_2 < N' < N_0$ or $N_1 < 1 < N' < N_0 < N_2$.

But

$M^2 > \sigma + 1$ cannot be satisfied for $N_1 < 1 < N' < N_0 < N_2$

for the following reasons. If this were possible, then from Eq. (7)

$H(N_1) = H(N_2) = H(N_3) = C$.

Then would exist $N''$ such that $H'(N'') = 0$ and $N_0 < N'' < N_2$.

Therefore, by the equation (7),

$F'(N'') = 0$ or $G(N'') = 0$,

i.e., either $F'(N'') = 0$ or $G(N'') = 0$, which contradicts the Observations 1 and 2. Hence, for the existence of a physical solution of (6), one must have

$N_1 < 1 < N_2 < N'$.

**Theorem 2**. For every $N_2$ such that $1 < N_2 < N'$, there exists a unique physical solution of Eq. (6) whose (strict) upper bound is $N_2$ and (strict) lower bound is $N_1$ determined by the equation

$$\int_{N_1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,.$$

**Proof**. From (14)

$$F'(N)G(N) < 0 \quad \text{for} \quad 1 < N < N', \qquad\qquad F'(N)G(N) > 0, \quad \text{for} \quad N < 1,$$

and we observe that

$$\int_{1}^{N_2} F'(N)G(N)\mathrm{d}N \quad \text{is finite, while} \quad \int_{0}^{1} F'(N)G(N)\mathrm{d}N \to \infty.$$

Then for an arbitrary $N_2$, we can choose a unique $N_1$ such that

$$\int_{N_1}^{1} F'(N)G(N)\mathrm{d}N = -\int_{1}^{N_2} F'(N)G(N)\mathrm{d}N\,,$$

i.e.,

$$\int_{N_1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,. \tag{15}$$

Therefore, from (6), (7) and (15), one gets

$$H(N_1) = H(N_2) = C \quad \text{and} \quad H(N) \geq C \quad \text{for} \quad N1 < N < N_2 \,.$$

Therefore, there exists a unique solution of Eq. (6) whose (strict) upper bound is $N_2$ $(1 < N_2 < N')$ and (strict) lower bound is $N_1$ determined by (15).

**Case 1b**.

$$M^2 > \sigma + 1, \quad 1 < N_0 < N' \,. \tag{16}$$

**Theorem 3**. For $M^2 > \sigma + 1$ and $1 < N_0 < N'$, the existence of a physical solution of Eq. (6) requires $N_1 < 1 < N_0 < N' < N_2$, where for given $M$, $\sigma$ and $\gamma > 0$, one can uniquely determine $N_1$ and $N_2$ by the equations

$$\int\limits_{N_1}^{N_0} F'(N)G(N)\mathrm{d}N = 0 \quad \text{and} \quad \int\limits_{N_0}^{N_2} F'(N)G(N)\mathrm{d}N = 0 \,.$$

Proof. From (12), (13) and (16), one gets

$$\begin{aligned} &F'(N)G(N) > 0 \ \text{ for } \ N < 1, && F'(N)G(N) < 0 \ \text{ for } \ 1 < N < N_0, \\ &F'(N)G(N) > 0 \ \text{ for } \ N_0 < N < N', && F'(N)G(N) < 0 \ \text{ for } \ N > N'. \end{aligned} \tag{17}$$

Therefore, from (9), (11), (16) and (17), it follows that

$$M^2 > \sigma + 1 \ \text{ for either } \ N_1 < 1 < N_2 < N_0 < N' \quad \text{or} \quad N_1 < 1 < N_0 < N' < N_2.$$

But $M^2 > \sigma + 1$ for $N_1 < 1 < N_2 < N_0 < N'$ cannot be satisfied for the following reasons. If it were possible, then from Eq. (7)

$$H(N_2) = H(N_0) = C \,.$$

Then $N''$ would be such that $H'(N'') = 0$ and $N_2 < N'' < N_0$.
Therefore, by Eq. (7), $F'(N'')G(N'') = 0$, i.e., either $F'(N'') = 0$ or $G(N'') = 0$, which contradicts the Observations 1 and 2. Since from (17),

$$F'(N)G(N) > 0, \ \text{ for } \ 0 < N < 1, \qquad F'(N)G(N) < 0, \ \text{ for } \ 1 < N < N_0,$$

and we observe that

$$\int\limits_{1}^{N_0} F'(N)G(N)\mathrm{d}N \qquad \text{is finite, and} \qquad \int\limits_{0}^{1} F'(N)G(N)\mathrm{d}N \to \infty \,.$$

Then one can choose $N_1$ uniquely such that

$$\int_{N_1}^{1} F'(N)G(N)\mathrm{d}N = -\int_{1}^{N_0} F'(N)G(N)\mathrm{d}N = 0\,,$$

i.e.,

$$\int_{N_1}^{N_0} F'(N)G(N)\mathrm{d}N = 0\,. \tag{18}$$

Again, since from (17) $F'(N)G(N) > 0$ for $N_0 < N < N'$ and $F'(N)G(N) < 0$ for $N > N'$, and we see that $\int_{N_0}^{N'} F'(N)G(N)\mathrm{d}N$ is finite and $\int_{N'}^{\infty} F'(N)G(N)\mathrm{d}N \to \infty$. Then one can choose $N_2$ uniquely such that

$$\int_{N_0}^{N'} F'(N)G(N)\mathrm{d}N = -\int_{N'}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,,$$

i.e.,

$$\int_{N_0}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,. \tag{19}$$

Hence the existence of physical solution of the equation (6) requires

$$N_1 < 1 < N_0 < N' < N_2\,,$$

where $N_1$ and $N_2$ are uniquely determined by (18) and (19).

**Case 2a**.

$$M^2 < \gamma^2(\sigma+1) \quad \text{for} \quad N_0 < N' < N_1\,. \tag{20}$$

**Theorem 4**. For $M^2 < \gamma^2(\sigma + 1)$ and $N_0 < N' < 1$, the existence of a physical solution of Eq. (6) requires $N' < N_1 < 1 < N_2$.

**Proof**. Since $G(1) = G(N') = 0$, we get from (2)

$$G(N) < 0 \;\text{ for }\; N < N', \quad G(N) > 0 \;\text{ for }\; N' < N < 1, \quad G(N) < 0 \;\text{ for }\; N \geq 1, \tag{21}$$

From (12) and (21), one gets

$$\begin{aligned} &F'(N)G(N) > 0 \;\text{ for }\; N < N_0, && F'(N)G(N) < 0 \;\text{ for }\; N_0 < N < N', \\ &F'(N)G(N) > 0 \;\text{ for }\; N' < N < 1, && F'(N)G(N) < 0 \;\text{ for }\; N > 1. \end{aligned} \tag{22}$$

From (9), (11), (20) and (22), it follows that

$M^2 < \gamma^2(\sigma+1)$ for either $N_1 < N_0 < N' < 1 < N_2$, or $N_0 < N' < N_1 < 1 < N_2$.

But $M^2 < \gamma^2(\sigma + 1)$ for $N_1 < N_0 < N' < 1 < N_2$ is impossible for the following reasons. If the above case were possible, then from (7)

$$H(N_1) = H(N_0) = H(N_2) = C\,.$$

Then there would exist $N^{IV}$ such that $H'(N^{IV}) = 0$ and $N_1 < N^{IV} < N_0$.

Therefore, by Eq. (7), $F'(N^{IV})G(N^{IV}) = 0$, i.e., either $F'(N^{IV}) = 0$ or, $G(N^{IV}) = 0$, which contradicts the Observations 1 and 2. Hence, the existence of physical solution of the equation (6) requires $N' < N_1 < 1 < N_2$.

**Theorem 5**. For every $N_1$ such that $N' < N_1 < 1$, there exists a unique physical solution of Eq. (6) whose (strict) lower bound is $N_1$ and (strict) upper bound is $N_2$ determined by the equation

$$\int\limits_{N_1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,.$$

**Proof**. Since from (22),

$$F'(N)G(N) > 0, \quad \text{for} \quad N' < N < 1 \qquad F'(N)G(N) < 0, \quad \text{for} \quad N > 1\,,$$

we observe that

$$\int\limits_{N_1}^{1} F'(N)G(N)\mathrm{d}N \quad \text{is finite} \quad \text{and} \quad \int\limits_{1}^{\infty} F'(N)G(N)\mathrm{d}N \to \infty\,.$$

Then there exists a unique $N_2$ such that

$$\int\limits_{1}^{N_2} F'(N)G(N)\mathrm{d}N = -\int\limits_{N_1}^{1} F'(N)G(N)\mathrm{d}N,$$

i.e.,

$$\int\limits_{N_1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,. \tag{23}$$

Therefore, from (6), (7) and (23), one gets

$$H(N_1) = H(N_2) = C, \qquad H(N) \geq C \quad \text{for} \quad N_1 < N < N_2\,.$$

Hence, there exists a unique solution of Eq. (6) whose (strict) lower bound is $N_1$ ($N' < N_1 < 1$) and (strict) upper bound is $N_2$ determined by Eq. (23).

**Case 2b**.

$$M^2 < \gamma^2(\sigma+1), \quad N' < N_0 < 1\,. \tag{24}$$

**Theorem 6**. For $M^2 < \gamma^2(\sigma+1)$ and $N' < N_0 < 1$, the existence of a physical solution of Eq. (6) requires $N_1 < N' < N_0 < 1 < N_2$ where for given $M$, $\sigma$ and $\gamma > 0$, one can uniquely determine $N_1$ and $N_2$ by the equations

$$\int\limits_{N_1}^{N_0} F'(N)G(N)\mathrm{d}N = 0 \quad \text{and} \quad \int\limits_{N_0}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,.$$

From (12), (21) and (24), one gets

$$\begin{aligned} &F'(N)G(N) > 0 \ \text{ for } \ N < N', \qquad && F'(N)G(N) < 0 \ \text{ for } \ N' < N < N_0, \\ &F'(N)G(N) > 0 \ \text{ for } \ N_0 < N < 1, && F'(N)G(N) < 0 \ \text{ for } \ N > 1. \end{aligned} \tag{25}$$

Therefore, from (9), (11), (24) and (25), it follows that

$$M^2 < \gamma^2(\sigma{+}1) \ \text{ for either } \ N_1 < N' < N_0 < 1 < N_2, \quad \text{or } N' < N_1 < N_0 < 1 < N_2\,.$$

But $M^2 < \gamma^2(\sigma+1)$ for $N' < N_1 < N_0 < 1 < N_2$ is impossible for the following reasons. If the above case were possible, then from (7),

$$H(N_0) = H(N_1) = H(N_2) = C\,.$$

Therefore, suppose there exists $N_V$ such that $H'(N^V) = 0$ and $N_1 < N^V < N_0$.

Therefore, by (7), $F'(N^V)G(N^V) = 0$, i.e., either $F'(N^V) = 0$ or $G(N^V) = 0$, which contradicts the Observations 1 and 2.

Since from (25),

$$F'(N)G(N) < 0 \ \text{ for } \ N' < N < N_0, \qquad F'(N)G(N) > 0, \ \text{ for } \ 0 < N < N'\,,$$

and one observes that

$$\int\limits_{N_1}^{N_0} F'(N)G(N)\mathrm{d}N \ \text{ is finite } \quad \text{and} \quad \int\limits_{0}^{N'} F'(N)G(N)\mathrm{d}N \to \infty\,.$$

Then there exists a unique $N_i$ such that

$$\int\limits_{N_1}^{N'} F'(N)G(N)\mathrm{d}N = -\int\limits_{N'}^{N_0} F'(N)G(N)\mathrm{d}N\,,$$

i.e.,

$$\int_{N_1}^{N_0} F'(N)G(N)\mathrm{d}N = 0\,. \tag{26}$$

Again, since from (25)

$$F'(N)G(N) > 0 \text{ for } N_0 < N < 1 \quad \text{and} \quad F'(N)G(N) < 0 \text{ for } N > 1\,,$$

and our observation is that

$$\int_{N_0}^{1} F'(N)G(N)\mathrm{d}N \text{ is finite} \quad \text{and} \quad \int_{1}^{\infty} F'(N)G(N)\mathrm{d}N \to \infty\,.$$

Then one can choose a unique $N_2$ such that

$$\int_{1}^{N_2} F'(N)G(N)\mathrm{d}N = -\int_{N_0}^{N_2} F'(N)G(N)\mathrm{d}N\,,$$

i.e.,

$$\int_{N_0}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,. \tag{27}$$

Hence, the existence of physical solution of (6) requires

$$N_1 < N' < N_0 < 1 < N_2\,,$$

where $N_1$ and $N_2$ are uniquely determined by Eqs. (26) and (27).

**Case 3**.

$$\sigma + 1 > M^2 > \gamma^2(\sigma + 1) \text{ for } N_0 < 1 < N'. \tag{28}$$

**Theorem 7**. For $\sigma + 1 > M^2 > \gamma^2(\sigma + 1)$ for $N_0 < 1 < N'$, the existence of a physical solution of Eq. (6) requires

$$N_0 < N_1 = 1 < N' < N_2\,.$$

Since $G(1) = G(N') = 0$, we get from (5)

$$\begin{aligned} &G(N) < 0 \text{ for } N < N_0, && G(N) \le 0 \text{ for } N_0 < N < 1, \\ &G(N) \ge 0 \text{ for } 1 < N < N', && G(N) < 0 \text{ for } N > N'. \end{aligned} \tag{29}$$

From (14) and (29), one gets

$$F'(N)G(N) > 0 \ \text{ for } \ N < N_0, \qquad F'(N)G(N) \le 0 \ \text{ for } \ N_0 < N < 1, \tag{30}$$
$$F'(N)G(N) \ge 0 \ \text{ for } \ 1 < N < N', \qquad F'(N)G(N) < 0 \ \text{ for } \ N > N'.$$

From (9), (11), (28) and (30), it follows that

$$\sigma + 1 > M^2 > \gamma^2(\sigma+1) \ \text{ for } \ N_0 < N_1 = 1 < N' < N_2\,,$$

which is the requirement for the existence of the physical solution of Eq. (6).

**Theorem 8**. For every $N_1 = 1$ such that there exists a unique physical solution of Eq. (6) whose (strict) lower bound is 1 and (strict) upper bound is $N_2$ determined by the equation (nešto fali)

$$\int\limits_{N_1}^{N_2} F'(N)G(N)\mathrm{d}N = 0.$$

**Proof**. Since from (30) one has

$$F'(N)G(N) > 0, \ \text{ for } \ 1 < N < N', \qquad F'(N)G(N) < 0, \ \text{ for } \ N > N'\,,$$

and we observe that

$$\int\limits_{1}^{N'} F'(N)G(N)\mathrm{d}N \ \text{ is finite } \ \text{ and } \ \int\limits_{N'}^{\infty} F'(N)G(N)\mathrm{d}N \to \infty.$$

Then one can choose a unique $N_2$ such that

$$\int\limits_{N'}^{N_2} F'(N)G(N)\mathrm{d}N = -\int\limits_{1}^{N'} F'(N)G(N)\mathrm{d}N\,,$$

i.e.,

$$\int\limits_{1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,. \tag{31}$$

From (6), (7) and (31), one gets

$$H(1) = H(N_2) = C \quad \text{and} \quad H(N) > C \ \text{ for } \ 1 < N < N_2\,.$$

Hence there exists a unique physical solution of Eq. (6) whose (strict) lower bound is 1 and (strict) upper bound is $N_2$ determined by (31).

## 4. *Analysis of the solutions*

For $M^2 > 1 + \sigma$, we obtained in the previous section two types of solution (solutions corresponding to ion-cyclotron waves) given in the Case 1a and Case 1b. Yashvir et al. [31] found only the solution given in the Case 1a. They showed that the solution exists only for the values of $E_0 \simeq 0.1 - 0.4$ where $E_0$ is the normalized electric field. Also, they concluded that $E_0$ takes neither very small nor very large values for the existence of ion-cyclotron waves. In the case of solution given in the Case 3 (solution corresponding to ion-acoustic solitons) they graphically showed the following results:

$$\text{(a)}\ \frac{\mathrm{d}N'}{\mathrm{d}\sigma} < 0, \qquad \text{(b)}\ \frac{\mathrm{d}N_2}{\mathrm{d}\sigma} < 0, \qquad \text{(c)}\ \frac{\mathrm{d}N_2}{\mathrm{d}M} > 0,$$

taking only few numerical values of $\sigma$ ($> 0$), $M$ ($> 0$) and $\gamma$ ($> 0$), and $N'$ and $N_2$ as functions of $\sigma$ and $M$.

In this context we present more rigorous analytical study with two following theorems.

**Theorem 9**. For given $\sigma > 0$, $M > 0$ and $M^2 > 1+\sigma$, ion-cyclotron waves can exist for arbitrarily small as well as arbitrarily large values of $E_0$.

**Proof**. Since $N_2$ and $N_1$ be the least upper bound and greatest lower bound of $N(s)$ and $(\mathrm{d}N/\mathrm{d}s)_{N=1}$, we get from (6) and (7)

$$\int\limits_{N_1}^{1} F'(N)G(N)\mathrm{d}N + \int\limits_{1}^{N_2} F'(N)G(N)\mathrm{d}N = 0\,, \tag{32}$$

and

$$\int\limits_{N_1}^{1} F'(N)G(N)\mathrm{d}N = (1+\sigma-M^2)E_0\,. \tag{33}$$

Since $(1 + \sigma - M^2)$ is fixed, by choosing $N_1 \simeq 1$, one gets from (33) that $E_0$ can have an arbitrarily small value.

Also, from (33) one gets arbitrarily large $E_0$ by choosing $N_1 \simeq 1$. Once $\int\limits_{N_1}^{1} F'(N)G(N)\mathrm{d}N$ is fixed, one can choose for $N_2$ any small and also any large value at will.

**Theorem 10**. For $\sigma + 1 > M^2 > \gamma^2(\sigma + 1)$ and $N_0 < N_1 = 1 < N' < N_2$,

$$(i)\ \frac{\mathrm{d}N'}{\mathrm{d}\sigma} < 0 \ \text{ for all } \ \sigma > 0, \qquad (ii)\ \frac{\mathrm{d}N_2}{\mathrm{d}\sigma} < 0 \ \text{ for all } \ \sigma > 0, \ \text{ and}$$

$$(iii)\ \frac{\mathrm{d}N_2}{\mathrm{d}M} > 0 \ \text{ for all } \ M > 0.$$

**Proof**. (i) Since $G(N) = 0$ at $N = N'(\neq 1)$, we get

$$N'^3 + N'^2 + \left(1 + \frac{3}{\sigma}\right)N' - \frac{3M^2}{\gamma^2\sigma} = 0\,. \tag{34}$$

Differentiating (34) with respect to $\sigma$, we get

$$\left[3N'^2 + 2N' + \left(1 + \frac{3}{\sigma}\right)\right]\frac{\mathrm{d}N'}{\mathrm{d}\sigma} = \frac{3}{\sigma^2}\left(N' - \frac{M^2}{\gamma^2}\right). \tag{35}$$

For $\sigma > 0$ and $N' > 0$, it follows from (35)

$$\frac{\mathrm{d}N'}{\mathrm{d}\sigma} < 0\,, \tag{36}$$

if

$$N' < \frac{M^2}{\gamma^2}\,. \tag{37}$$

The inequality (37) is obvious, since by combining (34) and (37), we get

$$N'(N'^2 + N' + 1)\sigma > 0\,,$$

which is true for all values of $N'$ ($> 0$) and $\sigma$ ($> 0$).

Hence, $\mathrm{d}N'/\mathrm{d}\sigma < 0$ for all $\sigma > 0$.

(ii) From (7) one gets

$$H(N_2,\sigma) = 2\left[\int_1^{N_2} u(N)\nu(N)\mathrm{d}N + \int_1^{N_2}(N\nu(N) - u(N)w(N))\mathrm{d}N - \sigma^2\int_1^{N_2} Nw(N)\mathrm{d}N\right], \tag{38}$$

where

$$G(N,\sigma) = \nu(N) - \sigma w(N), \quad F'(N,\sigma) = u(N) + \sigma N, \quad u(N) = -\frac{1}{N^3}(N^2 - M^2),$$

$$\nu(N) = (N-1)(M^2 - \gamma^2 N), \qquad w(N) = \frac{\gamma^2 N}{3}(N^3 - 1)\,.$$

Differentiating (38), we get

$$\frac{\partial H}{\partial N_2}\frac{\mathrm{d}N_2}{\mathrm{d}\sigma} + \frac{\partial H}{\partial \sigma} = 0\,. \tag{39}$$

From (7) and by observing that $G(N_2) < 0$, $F'(N_2) > 0$, we get

$$\frac{\partial H}{\partial N_2} < 0\,. \tag{40}$$

Differentiating (38) with respect to $\sigma$, we get

$$\frac{\partial H}{\partial \sigma} = 2\left[\int\limits_1^{N'} N(\nu - \sigma w)\mathrm{d}N + \int\limits_{N'}^{N_2} N(\nu - \sigma w)\mathrm{d}N - \int\limits_1^{N_2} (u + \sigma N) w \mathrm{d}N\right]. \tag{41}$$

Since $u + \nu N = F'(N) > 0$, for $1 < N < N_2$, $w = 1/(3\gamma^2 N)\,(N^3 - 1) > 0$ for $N > 1$, and $\nu - \sigma w = G(N) < 0$ for $N' < N < N_2$, we get

$$\int\limits_1^{N_2} (u + \sigma N) w \mathrm{d}N > 0 \qquad \text{and} \qquad \int\limits_{N'}^{N_2} N(\nu - \sigma w)\mathrm{d}N < 0\,.$$

Therefore, from (41) follows that $\partial H/\partial \sigma < 0$ will be valid if we prove that $\int\limits_1^{N'} N(\nu - \sigma w)\mathrm{d}N < 0$. It is clear from (41) that $\partial H/\partial\sigma$ will be of the same sign if we can prove

$$\int\limits_1^{N'} N\nu \mathrm{d}N - \sigma \int\limits_1^{N'} N w \mathrm{d}N - \int\limits_1^{N'} (u + N\sigma) w \mathrm{d}N < 0\,.$$

But $\int\limits_1^{N'} N\nu\mathrm{d}N < 0$, when $\nu(N) < 0$ for $1 < N < N'$, $\int\limits_1^{N'} N w \mathrm{d}N > 0$ when $w(N) > 0$ for $N > 1$, and $\int\limits_1^{N'} (u + N\sigma) w \mathrm{d}N > 0$ when $u + N\sigma = F')N) > 0$ for $1 < N < N'$.

$$\therefore \frac{\partial H}{\partial \sigma} < 0\,. \tag{42}$$

Therefore, from (39), (40) and (42), we get $\mathrm{d}N_2/\mathrm{d}\sigma < 0$.

(iii) From (7), one gets $H(N_2) = 2\int\limits_1^{N_2} F'(N)G(N)\mathrm{d}N = 0$, which can be written as

$$M^4 p(N_2) - M^2 q(N_2) + r(N_2) = 0\,, \tag{43}$$

where

$$p(N_2) = \frac{(N_2 - 1)^2}{2N_2^2}\,,$$

$$q(N_2) = \int_1^{N_2} \Big[ \Big( \frac{1}{N} + \sigma N \Big) + \frac{\gamma^2}{N^2} \Big\{ 1 + \frac{\sigma}{3} (N^2 + N + 1 \Big\} \Big] (N-1) \mathrm{d}N \,. \tag{44}$$

Differentiating (43) with respect to $M$, we get

$$4M^3 p - 2Mq + [M^4 p' - M^2 q' + r'] \frac{\mathrm{d}N_2}{\mathrm{d}M} - 0 \,. \tag{45}$$

Since $M^4 p' - M^2 q' + r' = -F'(N_2)G(N_2)$ and $F'(N_2)G(N_2) < 0$, (by (30)), we get

$$M^4 p' - M^2 q' + r' > 0 \,. \tag{46}$$

Equation (43) is a quadratic equation in $M^2$. The roots of the equation are given by

$$M^2 = \frac{q \pm \sqrt{q^2 - 4pr}}{2p} \,.$$

Since $p > 0$ (from (44)), $2pM^2 - q > 0$ or $< 0$ for larger or smaller value, respectively, of $M^2$ according to (43).

Now we examine which among the two roots is suitable for the physical need of our problem. The product of the roots of Eq. (43) is

$$\gamma^2 N_2^2 \Big[ 1 + \frac{\sigma}{3} (N_2^2 + N_2 + 1) \Big]^2 , \qquad G(N_2) < 0, \qquad \text{and} \;\; N_2 > 1 \,,$$

$$\therefore M^2 < \gamma^2 N_2 \Big[ 1 + \frac{\sigma}{3} (N_2^2 + N_2 + 1) \Big] < \gamma N_2 \Big[ 1 + \frac{\sigma}{3} (N_2^2 + N_2 + 1) \Big] \quad (\text{since } \gamma < 1),$$

i.e., the product of the roots of Eq. (43) is greater or equal to the square of a root of the equation, which is possible when the smaller root of the equation satisfies

$$\therefore 2pM^2 - q < 0 \,. \tag{47}$$

Therefore, from (45), (46) and (47) follows that $\mathrm{d}N_2/\mathrm{d}M > 0$ for all $M > 0$.

## 5. *Summary and concluding remarks*

In our present investigation, we have analytically studied the nonlinear propagation of finite-amplitude ion-cyclotron waves, ion-acoustic waves and solitons in a collisionless magnetized plasma consisting of warm ions and isothermal electrons and have obtained five types of solution (given in Case 1a, Case 1b, Case 2a and Case 2b of Sect. 3) for the existence of these waves. Two solutions given in Case 1a ($N_1 < 1 < N_2 < N' < N_0$) and Case 1b ($N_1 < 1 < N_0 < N' < N_2$) correspond to finite-amplitude ion-cyclotron waves, the two solutions given in Case

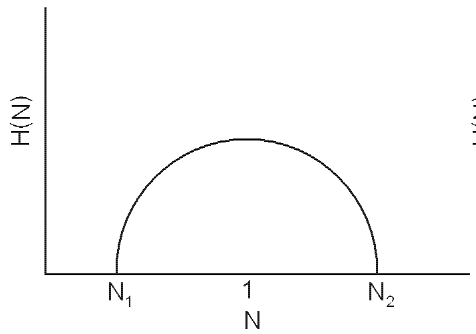
2a ($N_0 < N' < N1 < 1 < N_2$) and Case 2b ($N_1 < N' < N_0 < 1 < N_2$) correspond to finite-amplitude ion-acoustic waves and the solution given in Case 3 ($N_0 < N1 = 1 < N' < N_2$) correspond to ion-acoustic solitons. Our solutions given in Case 1a, Case 2a, and Case 3 have been also obtained by Yashvir et al. [31]. It is to be noted that solutions given in Case 1b and Case 2b are two new types of solution. The nature of the solutions for ion-cyclotron waves are represented graphically in Fig. 1 and Fig. 3, for the ion-acoustic waves in Fig. 2 and Fig. 4 and for solitons in Fig. 5. For the existence of ion-acoustic solitons, we have shown rigorously in Theorem 10:

i) $\mathrm{d}N'/\mathrm{d}\sigma < 0$ for all $\sigma > 0$,

ii) $\mathrm{d}N_2/\mathrm{d}\sigma < 0$ for all $\sigma > 0$, and

iii) $\mathrm{d}N_2/\mathrm{d}M < 0$ for all $M > 0$.

In this regard, it is important to note that Yashvir et al. [31] indicated that ion-cyclotron waves do not exist for very small and also for very large values of normalized electric field $E_0$. But, our present study shows (in Theorem 9) that ion-cyclotron waves can exist for arbitrarily small as well as arbitrarily large values of $E_0$.



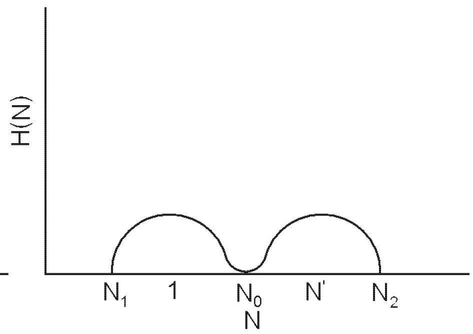




*Fig. 1. Plot of $H(N)$ versus $N$, ion-cyclotron wave ($M^2 > \sigma + 1,\ 1 < N' < N_0$).*
*Fig. 2. Plot of $H(N)$ versus $N$, ion-acoustic wave ($M^2 < \gamma^2(\sigma+1),\ N_0 < N' < 1$).*

However, in the present study, we have considered the plasma to be non-relativistic, collisionless and magnetized. But, the consideration of relativistic effect in plasma gives more interesting and fascinating results on nonlinear propagation of both electromagnetic and electrostatic waves in plasma [32 - 36]. Das and Paul [37] were the first to introduce relativistic term in K-dV equation for the study of ion-acoustic solitary waves in plasma. They showed that the relativistic effect would be introduced on the ion acoustic solitons only in the presence of streaming of ions in the plasma. Subsequently, Nejoh [38], Roy Chowdhury et al. [39], Paul et al. [40, 41] and other authors [42 - 45] investigated ion-acoustic solitary waves in various kinds of plasma considering negative ions, electron inertia, two-temperature electrons etc., and obtained important results. Following the mathematical tech-

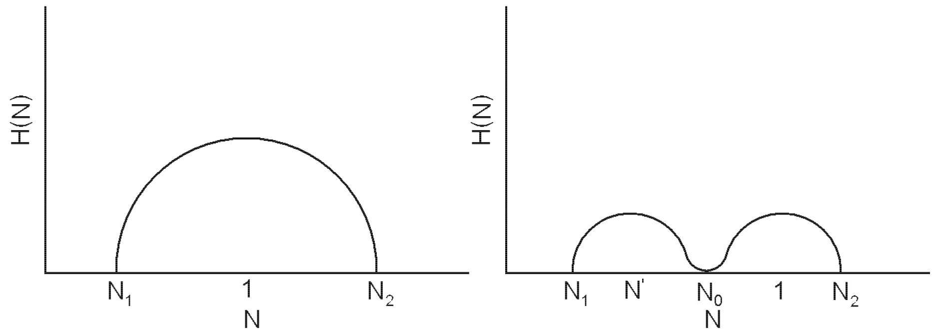



*Fig. 3. Plot of $H(N)$ versus $N$, ion-cyclotron wave (corresponding to the solution given in the Case 1b ($M^2 > \sigma + 1,\ 1 < N_0 < N'$).*

*Fig. 4. Plot of $H(N)$ versus $N$, ion-acoustic wave (corresponding to the solution given in the Case 2b ($M^2 < \gamma^2(\sigma + 1),\ N' < N_0 < 1$).*

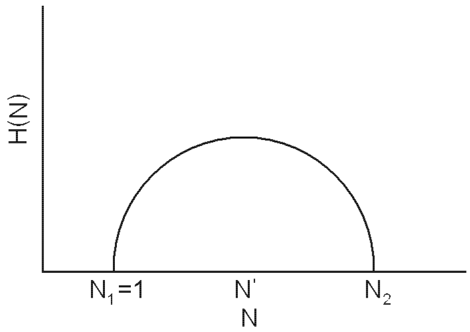



*Fig. 5. Plot of $H(N)$ versus $N$, soliton (corresponding to the solution given in the Case 3 ($\sigma + 1 > M^2 > \gamma^2(\sigma + 1),\ N_0 < 1 < N'$).*

nique in the present paper, some new analytical results may be obtained for both ion acoustic solitary waves and ion-cyclotron waves in a relativistic multicomponent plasma which would be applicable in various physical situations. The authors are exploring the possibility to explain some of the nonlinear phenomena observed in dusty plasma using the present theory.

## Appendix

**Proof of the Observation 1**. One can easily find from (5) that $G(1) = 0$ and the roots of the equation $G(N) = 0$ other than 1 are given by

$$g(N) = 0\,, \tag{48}$$

where,

$$g(N) = N^3 + N^2 + \Big(1 + \frac{3}{\sigma}\Big)N - \frac{3M^2}{\gamma^2\sigma}\,. \tag{49}$$

Since the physical constants $\sigma$, $\gamma$ and $M$ are positive, Eqn. (34) has only one positive root, say $N = N'$ while from (49) one gets $g(0) < 0$ and $g(\infty) > 0$.

$$\therefore g(1) < 0 \Rightarrow N' > 1 \qquad \text{and} \qquad g(1) > 0 \Rightarrow N' < 1 \,. \tag{50}$$

From (49) and (50), one gets

$$N' \gtrless 1 \text{according to} \gamma^2(\sigma+1) \gtrless M^2 .$$

**Proof of the Observation 2**. Solving the biquadratic equation $F'(N) = 0$, we get

$$N = \pm\sqrt{\frac{-1 \pm \sqrt{1+\sigma M^2}}{2\sigma}} \,.$$

Therefore, for $N > 0$, $N = +\sqrt{\dfrac{-1 + \sqrt{1+4\sigma M^2}}{2\sigma}}$ .

Therefore, for $N > 0$, $F'(N) = 0$ can hold at only one point $N = N_0$ (say).

Therefore, from (5) we get $\beta(N) = 0$ at $N = N_0$, where

$$\beta(N) \equiv \sigma N^4 + N^2 - M^2 \,. \tag{51}$$

Since $\sigma$ and $M(> 0)$ are constants, we get $\beta(0) < 0$ and $\beta(\infty) > 0$.

$$\therefore \beta(1) > 0 \Rightarrow N_0 < 1 \quad \text{and} \quad \beta(1) < 0 \Rightarrow N_0 > 1 \,. \tag{52}$$

From (51) and (52), one gets $\sigma + 1 \lesseqgtr M^2$ according to $N_0 \lesseqgtr 1$.

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## NELINEARNO ŠIRENJE VALOVA U TOPLOJ MAGNETIZIRANOJ PLAZMI

Teorijski proučavamo širenje ciklotronskih, ionsko-zvučnih i solitonskih valova male amplitude u magnetiziranoj plazmi bez sudara, koja se sastoji od toplih iona i izotermnih elektrona. Našli smo nova analitička rješenja za uzbudu ovih valova u plazmi. Grafički prikazujemo prirodu tih valova. Pokazuje se da ionsko-ciklotronski valovi mogu postojati za proizvoljno male kao i velike jakosti normaliziranog električnog polja, što nije suglasno s rezultatima drugih autora.