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ENERGY DEPENDENCE OF SELFSIMILARITY TRUNCATION IN A SYSTEM OF WEAKLY COUPLED DISSIPATIVE OSCILLATORS RELEVANT FOR BIOLOGICAL SYSTEMS

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Dedicated to Professor Kseno Ilakovac on the occasion of his 70$^{\mathbf{th}}$ birthday

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Biological fractals are truncated, i.e., their selfsimilarity extends at most over a few orders of magnitude of separation. In J. Theor. Biology 212 (2001) p. 47, we have shown that the nonlinear coupled oscillators, modeling one of the basic features of biological systems, may generate truncated fractals: a truncated fractal pattern for basin boundaries appears in a simple mathematical model of two coupled nonlinear oscillators with weak dissipation. We show here that the degree of truncation decreases with increasing energy. We point out that at the level of a sufficiently fine precision technique, the truncated fractality acts as a smooth (nonfractal) structure, leading to predictability, but at a lower level of precision it is effectively fractal, limiting the predictability of the long term behaviour of biological systems. Consequently, a possible erratic nature of the system's behaviour due to truncated fractality may disappear once the experimental errors in the measurement and/or treatment of biological system reaches a certain level of precision. We point out a possible significance of this result for the biological control of processes.

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1. Introduction

In a recent paper [1], we have proposed a possible new approach to gain insight into the fundamental question of the origin of truncated fractals in biological systems, which is a continuation of our former investigations related to the problem of chaotic transients and truncated fractals [2–12]. Two simple models for the symmetric and asymmetric system were studied in Ref. [1], showing the appearance of a truncation of selfsimilarity for coupled nonlinear dissipative oscillators. Here we investigate the energy dependence of truncation for the asymmetric model.

Significant progress has been made in physical and biological sciences due to discovery of fractal geometry of nature [13–15]. In biological and medical sciences, there is an increasing number of applications of both spatial and temporal fractal structures. Many biological phenomena appear to be fractal, for example the structure of bronchial tree [16], heartbeat dynamics [17–20], protein surfaces [21], chromatin microscopic images of breast epithelial cell nuclei [22], fetal breathing dynamics [23], microbial growth pattern [24], reduction law of metabolism [25], fetal heart rate [26], convoluted surface of mammalian brain [27], neural networks [28], urinary collecting tubes [28], long-range power-law correlation in DNA [29–31], neuronal shape [32], pattern in human retinal vessels [33], structure of biomembranes [34], blood vessel system [35,36], etc. In some of fractal biological phenomena, it is the spatial shape of a biological object itself that exhibits obvious fractal features, while in other cases the fractal properties are more hidden and can only be perceived if data are studied as a function of time or of some other variable, or mapped in some particular way [37], which is referred to as a hidden fractal property.

The reason why nature prefers fractal structures to those generated by classical scaling is that more effective function is achieved, but it may also be related to higher tolerance that fractal structures and processes possess over those of classical structures and processes [38]. It was argued that the fractal geometry may not only be a design principle for living organisms, but may also underlie an evolutionary advantage of biological systems having fractal dimension [39].

In all fractals, the underlying concept is selfsimilarity. In mathematical idealization, fractals are selfsimilar at all scales. This is a feature of exact mathematical fractals like, for example, Cantor set or Koch curve [13,40]. A similar feature appears also for fractal basin boundaries computed for mathematical models of dynamical systems [41–43], associated with multistability which is a fundamental property of nonlinear systems.

On the other hand, contrary to such mathematical fractals, for any real object in nature, statistical fractal properties are observed only over a limited size range. As pointed out by Mandelbrot, naturally fractal objects are statistically selfsimilar above some lower cutoff ϵ up to some upper cutoff value Ω [13]. Thus, the dimension of a naturally occurring fractal is associated with selfsimilarity over some region of space or interval of time. Therefore, such fractals are referred to as truncated fractals. On the basis of fractality of a wide range of natural systems, it was pointed out to the narrow range of appropriate scaling properties for declared fractal objects, centered around 1.3 orders of magnitude [44]. One of the most extensive fractal systems found in nature are the sedimentary rocks, formed from a mixture of organic and inorganic debris deposited in aqueous environment, with the range of length scales extended to over three decades [45].

In Ref. [1], we have addressed the following theoretical question: Can truncated fractals be generated in simple mathematical models, bearing some features of basic dynamics characterizing biological systems? As a guideline to identify such a simplified mathematical model, we turn to the well known observation that the key biological systems, such as, for example, cardiac, neural, respiratory, neuromuscular and hormonal, display intrinsic oscillatory behaviour [46–52]. Biological oscillators interact both with one another and with the environment. Moreover, there are innumerable feedback loops acting on physiological variables. Instigated by this observation, we have looked in Ref [1] for a possible origin of truncated biological fractals at the level of a simple mathematical model of coupled oscillators.

Nonlinear systems can produce fractal basin boundaries that separate basins of different attractors [41,43,53,54] and even more complicated examples of riddled and intermingled basins were found for specific dynamical systems [42,43,55]. In Ref. [1], we have considered fractal basin boundary as a hidden fractal property in the framework of biological systems, showing that the fractal basin boundary associated with dynamical systems of coupled oscillators, modeling some basic aspects of biological systems, can be truncated.

A possibility that the basin boundaries are truncated fractals was previously investigated for sinusoidally forced pendulum by introducing an additional exponential factor, so that the nonautonomous driving term exponentially decays to zero [56,57]. Moreover, in the case of a single Duffing oscillator, it was found that truncated fractal Arnold tongues can be finely intermingled with selfsimilar fractal Arnold tongues [10].

2. *Asymmetric system of coupled oscillators*

A simple model for asymmetric dissipative system is given by a one-well oscillator and a nonlinear double-well oscillator coupled by a linear interaction term, with equations of motion [1]

$$\begin{aligned} \ddot{x} + \gamma\dot{x} - x + x^3 + \alpha(x-y) &= 0 \\ \ddot{y} + \gamma\dot{y} + y - \alpha(x-y) &= 0\,, \end{aligned} \qquad (1)$$

where γ and α are the dissipation and coupling strength, respectively. (The linear term $\alpha(x-y)$ in the equations of motion corresponds to a quadratic term in the corresponding Hamiltonian.) The linear coupling between the two oscillators in the equations of motion was previously considered for the nonlinear mass-spring system [58] and in connection with scalar diffusion [59].

The system (1) has two attractors at positions of local minima of potential energy,

$$(x,y) = (x_{\min}, y_{\min}) \quad \text{and} \quad (-x_{\min}, -y_{\min})\,, \qquad (2)$$

where $x_{\min} = 1/\sqrt{1+\alpha}$ and $y_{\min} = \alpha/\sqrt{(1+\alpha)^3}$. In the graphical presentation for basin boundaries, we denote the initial conditions ending in the attrac-

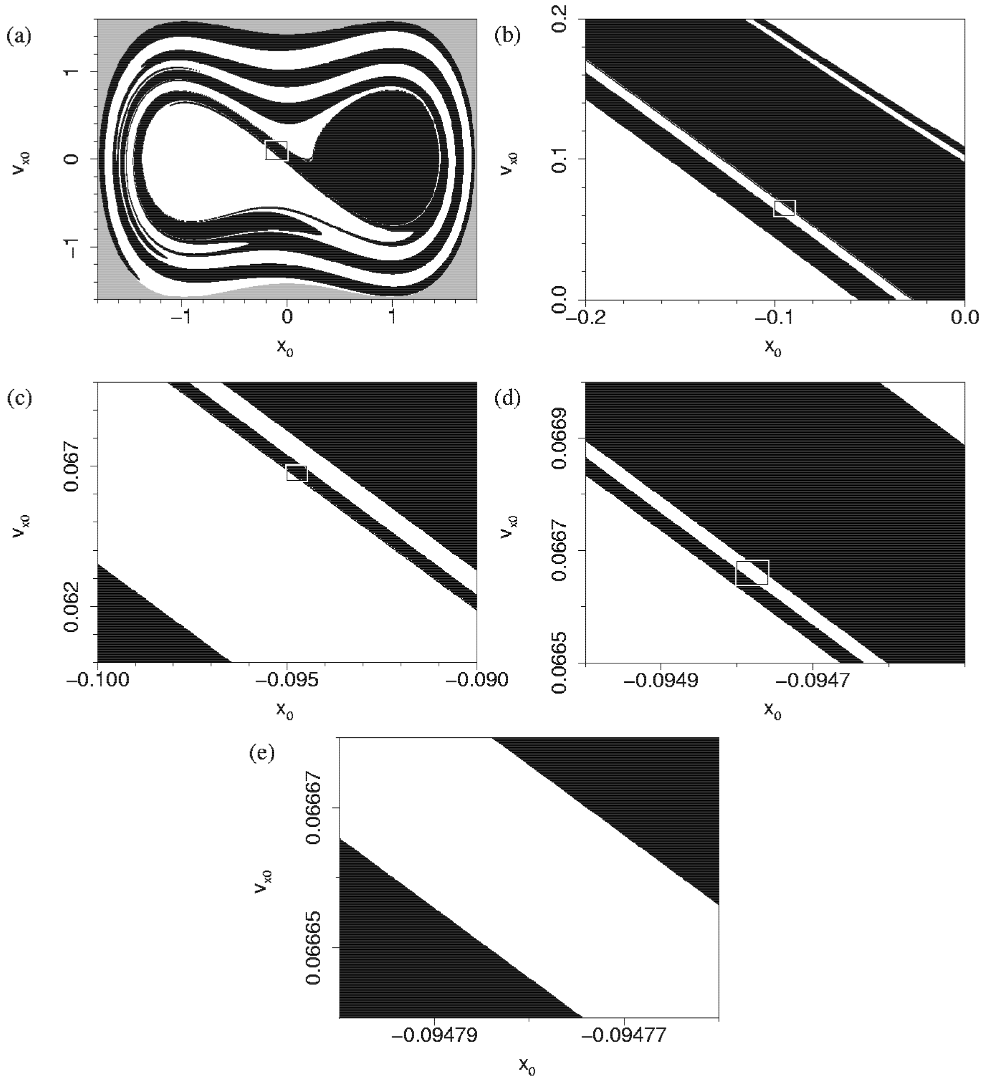

Fig. 1. Basins of attraction for the nonlinear system (1) with weak dissipation $\gamma = 0.05$ and weak coupling strength $\alpha = 0.05$ at the initial energy $E_0 = 1.0$. Four successively expanded views are shown in Figs. b – e. For description see the text.

tor $(-x_{\min}, -y_{\min})$ as black, while the initial conditions ending in the attractor $(x_{\min}, y_{\min})$ are left blank. On the other hand, the energetically inaccessible initial conditions are presented as grey. Initial values $(x_0, \dot{x}_0)$ are taken from a grid of resolution 570×400, while the initial values $(x_0, \dot{x}_0)$ are kept fixed. For all calculations, we take $y_0 = 0$ and $\dot{y}_0 \geq 0$ is determined from the condition that all initial conditions should lie on a plane defined by the initial energy [1].

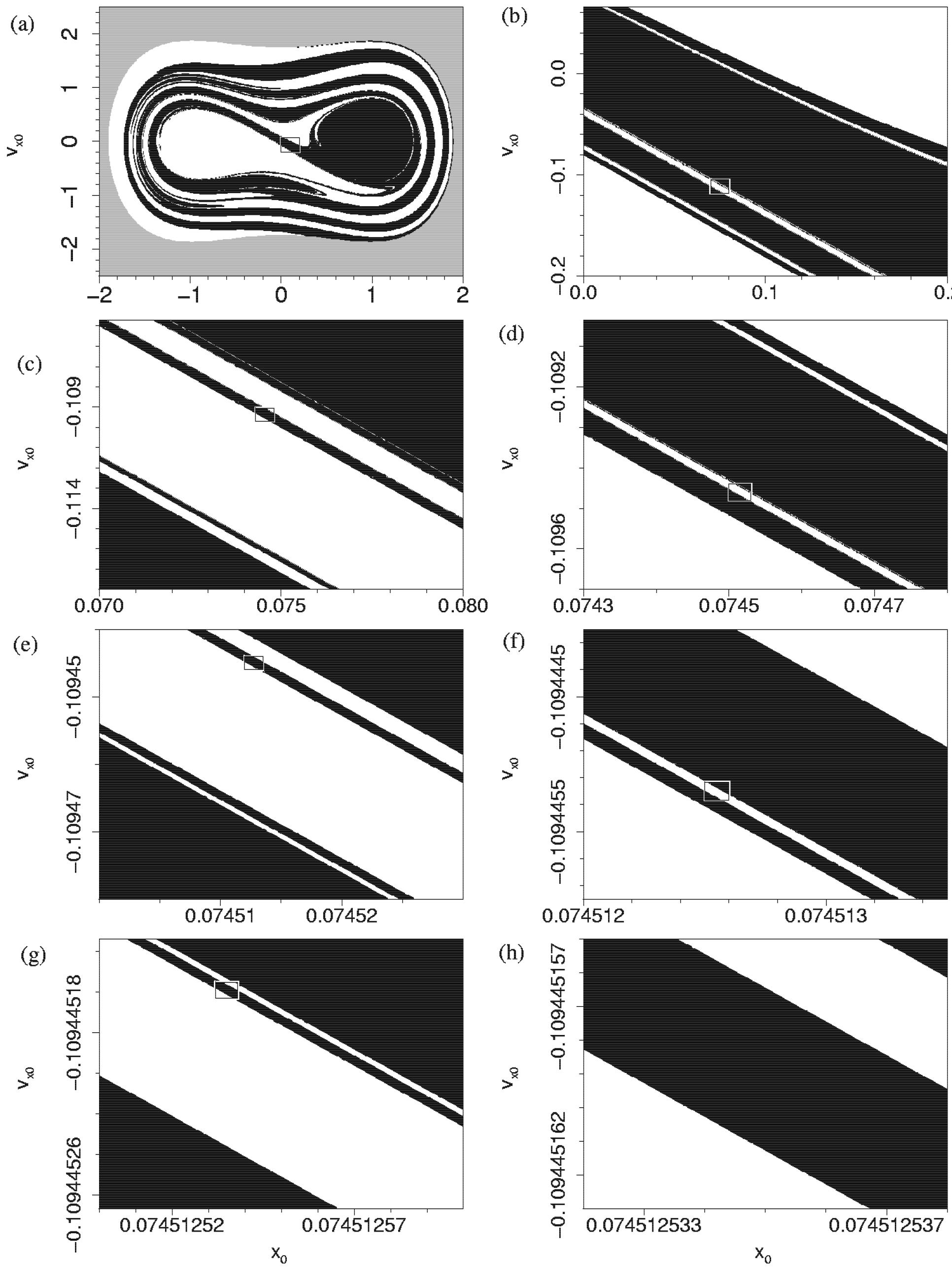

Fig. 2. Basins of attraction for the nonlinear system (1) with dissipation $\gamma = 0.05$ and coupling strength $\alpha = 0.05$ at the initial energy $E_0 = 1.5$. Seven successively expanded views are shown in Figs. b – h. For description see the text.

We have computed basin boundaries for the asymmetric system of coupled oscillators (1) looking for the energy dependence of the pattern of truncation. The coupling parameter α and the dissipation parameter γ used in this paper have sizeably lower values than used in Ref. [1], in order to get a more selfsimilar pattern revealing the energy dependence of the truncation effect on the initial energy. Similarly as in Ref. [1], the basins of attraction are calculated for a grid of 570×400 initial conditions $(x_0, \dot{x}_0)$ from the interval $-1.8 < x_0 < 1.8$, $-1.6 < \dot{x}_0 < 1.6$. Here, the calculations are performed for two different values of initial energy E_0, 1 and 1.5. For each case, successive magnifications are shown until the structure with smooth boundaries is reached. The results are displayed in Figs. 1 and 2, respectively.

Introduction of dissipation into the system leads to a decrease of effective nonlinearity with time. As time evolves, the energy of the system gradually decreases leading to truncation of the chain of stretchings and foldings. The larger is the initial energy, the sooner will stop the chain of successive stretchings and foldings. Our results show that for the energy of 1.0, the chain of repeated magnifications stops at a smooth pattern after three steps (Fig. 1), while for the energy of 1.5, the chain of repeated magnifications stops after six steps (Fig. 2). This is consistent with the statement from Ref. [1] that the fractals of truncated type remain as "shadows" of stationary chaos in the conservative system, additionally revealing a dependence on energy.

3. Discussion

In connection to the relevance of the present results for biological systems we note, similarly as in Ref. [1], that a nonlinear two-coupled-oscillator model was implemented in some recent studies of the dynamics of various biological systems, as for example for description of oscillations in two cyclin dependent protein kinases activity, involving mutual inhibitions of two oscillators and coexistence of various attractors [61], for simulation of calcium oscillations in leaves of Desmodium pulvini [62], for description of temporal organization of insects in the interior of caves regarding coupled moulting and oviposition cycles of Folsomia candida [63], for representation of a dopaminergic neuron as a set of electrically coupled oscillators [64], for the study of mode locking and Arnold tongues in the case of two synaptically coupled neural oscillators [65], for description of a cell patterning method with a plasmodial slime mold in which parameters of coupling can be systematically controlled [66], for the study of the cultured cells of cardiac pacemakers [67], for discussion of the circadian locomotor rhythm of nocturnal rodents in terms of regulation [68], for description of the human sleep-wake and body temperature rhythms [69], for a model of coupled oscillators employed as a model of the central pattern generator generating functional (also locomotional) rhythms [70], for interpretation of circadian rhythmicity in photoperiodic induction of diapause in a drosophilid fly Chymomyza costata [71], for a simple model of the sinoatrial node with cells being electrically coupled by linear conductance [72] and for a model for the study of heartbeat dynamics to describe the interaction between the sinoatrial and the atrioventricular node [73]. Presumably, many more examples of coupled biological oscillators will be found to play an important role in biological processes.

Furthermore, as pointed out in Ref. [1] and additionally elaborated here, a simple mechanism of coupled oscillators can lead to a complex coexistence of various modes involving a truncated fractal pattern, as truncated fractal basin boundaries and consequently the fractal boundaries in the parameter space. This fractality may play a role in generating some basic features of biological systems. On one hand, the appearance of fractality at a certain range of scale can be associated with a higher tolerance in physiological functions which is important for the adaptability of biological systems [39]. On the other hand, the appearance of truncation in the fractal pattern enables an appearance of a predictable long term behaviour of the system in conjunction with fractality once a certain level of precision in investigating and/or treating biological system has been achieved. Consequently, a possible erratic nature of the systems behaviour due to truncated fractality may disappear once the experimental errors in the measurement and/or treatment of biological system reaches a certain level of precision.

4. *Conclusion*

We point out that a model of coupled nonlinear oscillators with weak dissipation generates a truncated fractal pattern for basin boundaries, and consequently of boundaries in parameter space, which can be considered as a kind of hidden fractal property in the context of biological fractals.

It is clear that truncated fractals are more common in biological applications than the idealized case of true fractals. At a rougher precision technique truncated fractals can cause for practical purposes a similar obstruction to estimating the long term behaviour as true fractals, but on the other hand, at a sufficiently fine precision technique the predictability of the long term behaviour of biological systems is not fundamentally limited. This may give a new insight into the problem of truncated fractals in biological systems, which are characterized by a plethora of oscillatory behavior. In particular, this may shed new light on the problem of sensitive dependence on the type of therapy, basically limiting its efficacy in the clinical praxis. On the basis of our approach we are proposing an idea that a sufficient increase in precision of determining parameters involved in a particular therapy might lead to a regime of more deterministic and thus of more successful medical therapy in cases hindered due to effects associated with fractality. In order to test this idea, detailed clinical studies in relevant cases are desirable.

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ENERGIJSKA OVISNOST REZANJA SAMOSLIČNOSTI U SUSTAVU SLABO VEZANIH DISIPATIVNIH OSCILATORA OD ZNAČENJA ZA BIOLOŠKE SUSTAVE

Biološki oscilatori su odrezani, tj. njihova samosličnost se proteže kroz najviše nekoliko redova veličine. U radu J. Theor. Biology 212 (2001) p. 47 pokazali smo da nelinearni vezani oscilatori, koji modeliraju neka od bitnih svojstava bioloških sustava, mogu stvarati odrezane fraktale: odrezani fraktali za granice bazena pojavljuju se u jednostavnom matematičkom modelu dvaju vezanih nelinearnih oscilatora sa slabom disipacijom. Ovdje pokazujemo da se stupanj odrezanosti fraktala smanjuje s porastom energije. Ukazuje se da na razini dovoljno precizne tehnike odrezana fraktalnost djeluje kao glatka (nefraktalna) struktura, koja vodi na prediktibilnost, ali na razini manje preciznosti ponašanje je efektivno fraktalno, ograničujući mogućnost predviđanja dugoročnog ponašanja bioloških sustava. Kao posljedica, mogući eratični tip ponašanja sustava zbog odrezane fraktalnosti može iščeznuti kada eksperimentalne pogreške pri mjerenju i/ili tretmanu biološkog sustava dosegnu neku određenu razinu preciznosti. Naglašena je moguća važnost tog učinka za kontrolu bioloških procesa.